

# $D = 4, N = 2$ Supergravity in the Presence of Vector-Tensor Multiplets and the Role of higher $p$ -forms in the Framework of Free Differential Algebras

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## Abstract

We thoroughly analyze at the bosonic level, in the framework of Free Differential Algebras (FDA), the role of 2-form potentials setting in particular evidence the precise geometric formulation of the anti-Higgs mechanism giving mass to the tensors. We then construct the (super)-FDA encoding the coupling of vector-tensor multiplets in  $D = 4, N = 2$  supergravity, by solving the Bianchi identities in superspace and thus retrieving the full theory up to 3-fermions terms in the supersymmetry transformation laws, leaving the explicit construction of the Lagrangian to future work. We further explore the extension of the bosonic FDA in the presence of higher  $p$ -form potentials, focussing our attention to the particular case  $p = 3$ , which would occur in the construction of  $D = 5, N = 2$  supergravity where some of the scalars are properly dualized.

# 1 Introduction

The role of tensor multiplets in supergravity has seen in the last years a revived interest, in connection with the study of flux compactifications in superstring and M-theory. Indeed, it is well known that in string theory one should expect  $p$ -forms of various degree  $p$  to enter in the non-perturbative formulation of the theory, since they couple to extended objects ( $p$  and  $D$  branes).

The case of  $p = 2$  has been already discussed in the literature in various contexts. Two-index antisymmetric tensors are 2-form gauge fields whose field-strengths are invariant under the (tensor)-gauge transformation  $B \rightarrow B + d\Lambda$ ,  $\Lambda$  being any 1-form. A physical pattern to introduce massive tensor fields is the anti-Higgs mechanism, where the dynamics allows the tensor to take a mass by a suitable coupling to some vector field. The mass term plays the role of magnetic charge in the theory. The investigation of the role of massive tensor fields was particularly fruitful for the  $N = 2$  theory in 4 dimensions, where the study of the coupling of *scalar-tensor* multiplets (obtained by Hodge-dualizing scalars covered by derivatives in the hypermultiplet sector) in  $N = 2$  supergravity was considered, both as a CY compactification [1] and at a purely four dimensional supergravity level [2, 3]. When this model was extended, in [4, 5, 6], to include the coupling to gauge multiplets, it allowed to construct new gaugings containing also magnetic charges, and to find the electric/magnetic duality completion of the  $N = 2$  scalar potential. As far as the coupling of the *vector-tensor multiplets* is concerned, the situation appears well established in five dimensional supergravity, where a general formulation of the theory has been given [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

The corresponding coupling of *vector-tensor multiplets* in  $N = 2$ ,  $D = 4$  supergravity, which can be thought of as coming from dualization of the real or imaginary part of the vector multiplet scalars, can be studied by dimensional reduction from the  $D = 5$ ,  $N = 2$  theory [22, 21], thus catching important properties of the couplings and of the gauge structure. However being obtained by dimensional reduction this approach fails in giving the most general four dimensional theory.

Another important step towards the construction of this theory has been done in ref. [23] by a thorough analysis of the corresponding rigid supersymmetric theory in the framework of the embedding tensor formalism.

A complete formulation is however still missing, and in the present paper

we have filled this gap by working out the relevant superspace Bianchi Identities coming from the most general  $N = 2$ ,  $D = 4$  Free Differential Algebra (FDA) underlying the theory.

In the FDA approach a natural related issue is the coupling of higher degree  $p$ -form potentials in supergravity theories as they can be thought as non perturbative sectors of standard supergravities where Hodge dualization of some field has been operated. However supergravity formulations in  $D$  dimensions coupled to forms of order higher than two have not been considered in detail until now, even if some insight has been given for  $p = 3$  in the framework of maximally extended supergravity [23]. Indeed  $p$ -forms are gauge fields  $C^{(p)}$  whose field-strengths are invariant under the (tensor)-gauge transformation  $C^{(p)} \rightarrow C^{(p)} + d\Lambda^{(p-1)}$ ,  $\Lambda$  being any  $(p - 1)$ -form. Being a gauge field, a  $p$ -form has  $\binom{D-2}{p}$  physical degrees of freedom, so that their existence as propagating fields is limited to  $p = 2$  in four and to  $p = 3$  in five dimensions, but higher  $p$ -forms can naturally appear in higher dimensional supergravities. Note that their inclusion would give a tool for the study of the non-perturbative structure of M or string theory.

We will show that, similarly to the  $p = 2$  case, higher rank antisymmetric tensor fields may take mass via the Higgs mechanism <sup>1</sup>.

In the present paper we study in a general setting the structure of the FDA's since they give the natural framework for the analysis of theories including  $p$ -forms of any degree.

In the absence of supersymmetry, that is if we do not include the gravitino 1-form, the general structure of the FDA is independent on the space-time dimension  $D$ , the only restriction being that the maximal degree of the  $p$ -form potentials is  $D - 1$ . Instead, in the supersymmetric case the FDA structure depends heavily on the space-time dimension, according to the different properties of the spinor representations of  $SO(1, D - 1)$ . At the pure bosonic level the general structure of the FDA with forms of any degree is very complicate. However, if one restricts the degree of the forms to  $p = 1$

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<sup>1</sup>The name of anti-Higgs mechanism was first introduced when the self-duality in odd dimensions for massive 2-forms was discovered, since the degrees of freedom of the massive 2-form are obtained by absorbing those of a 1-form, which plays a passive role contrary to the active role played in ordinary Higgs mechanism where the vector absorbs a 0-form. However since the mechanism by which a  $p$ -form takes mass by absorbing the degrees of freedom of a  $(p - 1)$ -form gauge field is, as we shall show, the same for any  $p$ , it should be called generalized Higgs mechanism, but we will adopt for simplicity the same locution "Higgs mechanism" as for the 0-form case.

and  $p = 2$  the analysis can be done in full generality.

We therefore discuss in full detail first the case of  $p = 2$  both in the bosonic and in the supersymmetric case. Before applying our consideration to  $D = 4, N = 2$  supergravity in the presence of vector-tensor multiplets we further study at the pure bosonic level the inclusion of 3-forms in the FDA, leaving its supersymmetric completion to a future investigation.

As far as 2-forms are concerned, we will first study at the bosonic level and in full generality the algebraic structure which any theory coupled to 2-index tensors and gauge vectors is based on. This requires the extension of the notion of gauge algebra to that of FDA that naturally accommodates in a general algebraic structure the presence of  $p$ -forms ( $p > 1$ ) and gives a precise understanding of the Higgs mechanism through which the antisymmetric tensors become massive. The discussion will be completely general, and will not rely on the dimensions of space-time (apart from the obvious request  $D \geq 4$ , in order to have dynamical 2-forms) nor on supersymmetry.

The outcome of the analysis is that FDA approach allows to interpret the resulting structure in a general group-theoretical way which is not evident with other approaches. Even at the bosonic level, the analysis of the FDA involving gauge vectors and 2-forms charged under a subalgebra of the gauge algebra, gives a geometric insight into the structure of the physical theories it underlies. Indeed we find that the inclusion of charged tensors implies a deformation of the general gauge structure which can be precisely codified in terms of a deformation of the structure constants and couplings of the gauge group. Moreover, we obtain a precise algebraic understanding of the Higgs mechanism through which the 2-forms can become massive.

Given the general structure of the bosonic case, we can then proceed to the supersymmetric extension of the FDA underlying the  $D = 4, N = 2$  supergravity theory. This can be done in two directions: either we analyze the resulting  $D = 4, N = 2$  theory in the presence of scalar-tensor multiplets, or in the presence of vector-tensor multiplets. The former case has already been treated in ref. [4, 5]. As already announced we focus our analysis on the latter theory, thus obtaining its general formulation. Indeed the supersymmetrization of the FDA allows us to solve explicitly the Bianchi identities in superspace thus obtaining the full theory, since, as it is well known, such solution implies the knowledge of the supersymmetry transformation laws and the equations of motion.

The result of the analysis of the Bianchi identities shows the peculiar feature that a consistent description of the FDA requires the simultaneous pres-

ence of fields related by Hodge duality, that is, besides the electric potentials  $A_\mu^M$  and the antisymmetric tensors  $B_{M|\mu\nu}$ , also their Hodge-dual magnetic potentials  $A_{M|\mu}$  and scalars  $Y^M$ . Actually, the fields  $A_{M|\mu}$  and  $Y^M$  will be recognised as the Hodge duals of  $A_\mu^M$  and  $B_{M|\mu\nu}$  only after implementation of the Bianchi identities, that is on shell.<sup>2</sup> This peculiarity has two important consequences. First of all, in absence of vector multiplets, the scalar potential of  $N = 2$  supergravity coupled to vector-tensor multiplets is symplectic invariant. Secondly, the Kähler–Hodge structure of the  $\sigma$ -model, which describes the off-shell geometry of the theory, can be and actually is different from the on-shell geometry found after dualization. The study of the minima of the scalar potential requires the knowledge of the on-shell  $\sigma$ -model geometry. The construction of the Lagrangian in the rheonomic approach invariant under such transformations and giving (in a simpler way) the equations of motion is left to future work.

The construction of the  $N = 2$  supersymmetric theory in five dimensions can be done along the same lines, but since it reproduces the existing results in the literature, we do not give its explicit construction here. We limit ourselves to make some remarks on the origin of the self-duality of the tensors in terms of the Higgs mechanism giving mass to 2-forms which were originally massless.

As far as the inclusion of  $p$ -forms with  $p \geq 3$  is concerned, we have further analyzed in detail the FDA bosonic structure. Such inclusion is particularly interesting in five dimensional supergravity for the case  $p = 3$ , since a scalar is Hodge-dual to a 3-form.<sup>3</sup> In the standard  $D = 5$ ,  $N = 2$  supergravity, we have two possible kinds of scalars, namely the scalars of the vector multiplets and the ones in the hypermultiplets. When the vectors are dualized into 2-index tensors, we further have scalars in the tensor multiplets. In each case, any of these scalars can be dualized to 3-forms, originating new couplings pertaining to different non-perturbative sectors of the theory. The supersymmetrization of the relevant FDA in order to construct the corresponding supergravity theory is under investigation.

Finally, considering higher  $p$ -forms, we also propose a possible bosonic extension of the structure obtained for  $p = 3$  to any forms with  $p > 3$ , giving a completely consistent (bosonic) FDA.

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<sup>2</sup>This kind of approach was already pursued, at the lagrangian level and for the maximally extended  $D = 4$ ,  $N = 8$  theory, in [24].

<sup>3</sup>Note that  $p = 3$  can also play a role in  $D = 4$  since its field-strength can appear as a flux in the compactification from higher dimensions.

The paper is organized as follows:

In section 2 we study the general FDA describing the coupling of two-index antisymmetric tensor fields to non-abelian gauge vectors and show in detail, for the general case, how the Higgs mechanism giving mass to the 2-forms takes place.

In section 3 we include 3-forms in the FDA structure, discussing different possible cases and give a possible extension of the bosonic FDA including forms of any degree.

In section 4, we apply the formalism to the case of  $N = 2$  four dimensional supergravity coupled to vector-tensor multiplets. In our geometric approach the construction of the theory is obtained by solving the Bianchi identities in superspace. This allows us to find the supersymmetry transformations rules, the constraints on the scalar geometry which define the appropriate  $\sigma$ -model of the theory, the fermionic shifts and the scalar potential. As for the scalar-tensor theory we find that the contribution from the vector-tensor sector to the scalar potential is symplectic invariant.

In section 5 we make some remarks on the issue of self-duality in five dimensions for massive 2-forms.

The Appendices contain technical details.

Appendix A gives the constraints arising in the bosonic Bianchi identities for the FDA in the presence of 3-forms;

Appendix B describes the superspace solution of the Bianchi identities for  $D = 4$ ,  $N = 2$  vector-tensor theory;

Appendix C outlines the dualization procedure to obtain the vector-tensor  $\sigma$ -model metric after dualization;

Appendix D contains our conventions and notations.

## **2 A general bosonic theory with massive 2-index tensors and non-abelian vectors**

In this section we are going to study the gauge structure of a general theory with two-index antisymmetric tensor fields coupled to gauge vectors. The discussion here will be general, with no need to make reference to any particular dimension of space-time nor to any possible supersymmetric extension of the model. Later, in section 4, we will consider the supersymmetrization of the model, specifying the discussion to the case of four dimensional  $N = 2$

supergravity coupled to vector and vector-tensor multiplets.

## 2.1 FDA and the anti-Higgs mechanism

### 2.1.1 Abelian case

The simplest case of a FDA including 1-form and 2-form potentials <sup>4</sup> is described by a set of abelian gauge vectors  $A^M$  and of massless tensor two-forms  $B_M$  ( $M = 1, \dots, n_T$ .) interacting by a coupling  $m^{MN}$ . The field-strengths are:

$$\begin{cases} F^M &= dA^M + m^{MN} B_N \\ H_M &= dB_M \end{cases} \quad (2.1)$$

and are invariant under the gauge transformations:

$$\begin{cases} \delta A^M &= d\Theta^M - m^{MN} \Lambda_N \\ \delta B_M &= d\Lambda_M \end{cases} \quad (2.2)$$

with  $\Theta^M$  parameters of infinitesimal U(1) gauge transformations and  $\Lambda_M$  one-form parameters of infinitesimal tensor-gauge transformations of the two-forms  $B_M$ . In this case the system undergoes the Higgs mechanism, and it is possible to fix the tensor-gauge so that:

$$\begin{cases} A^M \rightarrow A'^M &= -m^{MN} \bar{\Lambda}_N \\ B_M \rightarrow B'_M &= B_M + d\bar{\Lambda}_M; \end{cases} \quad (2.3)$$

In this way the gauge vectors  $A^M$  disappear from the spectrum providing the degrees of freedom necessary for the tensors to acquire a mass, since:

$$\begin{cases} F'^M &= m^{MN} B_N \\ H'_M &= dB_M. \end{cases} \quad (2.4)$$

### 2.1.2 Coupling to a non-abelian algebra

The model outlined above may be generalized by including the coupling of this system to  $n_V$  gauge vectors  $A^X$  ( $X = 1, \dots, n_V$ ), with gauge algebra  $G_0$ ,

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<sup>4</sup>0-forms will also be included in section 4 when considering supersymmetric versions of the theory

if the index  $M$  of the tensors  $B_M$  and of the abelian vectors  $A^M$  runs over a representation of  $G_0$ . In this case the FDA becomes <sup>5</sup>:

$$\left\{ \begin{array}{lcl} F^X & = & dA^X + \frac{1}{2}f_{YZ}^X A^Y \wedge A^Z \\ F^M & = & dA^M - T_{XN}^M A^X \wedge A^N + m^{MN} B_N \\ & \equiv & DA^M + m^{MN} B_N \\ H_M & = & dB_M + T_{XM}^N A^X \wedge B_N + d_{XNM} F^X \wedge A^N \\ & \equiv & DB_M + d_{XNM} F^X \wedge A^N \end{array} \right. \quad (2.5)$$

Setting  $F^X = 0$  the first equations give the Cartan-Maurer equations of the Lie Algebra  $G_0$  dual to the formulation in terms of the generators  $T_X$  with structure constants  $f_{YZ}^X$ . Here  $T_{XM}^N$  and  $d_{XMN}$  are suitable couplings. The closure of the FDA ( $d^2 A^X = d^2 A^M = d^2 B_M = 0$ ) gives the following constraints:

$$f_{[XY}^W f_{Z]W}^\Omega = 0 \quad (2.6)$$

$$T_{[X|M}^P T_{Y]P}^N = \frac{1}{2} f_{XY}^Z T_{ZM}^N \quad (2.7)$$

$$T_{XM}^N = -d_{XMP} m^{NP} = d_{XPM} m^{PN} \quad (2.8)$$

$$T_{XN}^M m^{NP} = -T_{XN}^P m^{MN} \quad (2.9)$$

$$T_{XM}^P d_{YNP} + T_{XN}^P d_{YPM} - f_{XY}^Z d_{ZNM} = 0. \quad (2.10)$$

Eqs. (2.6), (2.7) show in particular that the structure constants  $f_{YZ}^X$  do indeed close the algebra  $G_0$  and that  $T_{XM}^N$  are generators of  $G_0$  in the representation spanned by the tensor fields. eqs. (2.8) and (2.9) imply:

$$\begin{aligned} m^{MN} &= \mp m^{NM} \\ d_{XMN} &= \pm d_{XNM}. \end{aligned} \quad (2.11)$$

Note that (2.10) is a consistency condition that, when multiplied by  $m^{PQ}$ , is equivalent to (2.7) (upon use of (2.9)). From the physical point of view it simply expresses the gauge covariance of the constants  $d_{XMN}$ , while from the geometric point of view equation (2.10) means that  $d_{XMN}$  are cocycles of

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<sup>5</sup>We will generally assume, here and in the following, that the tensor mass-matrix  $m^{MN}$  is invertible. In case it has some 0-eigenvalues, we will restrict to the submatrix with non-vanishing rank. This is not a restrictive assumption, because any tensor corresponding to a zero-eigenvalue of  $m$  may be dualized to a gauge vector and so included in the set of  $\{A^X\}$ .



the Lie Algebra Chevalley cohomology, in the given representation labelled by the indices  $MN$ .

When (2.6) - (2.10) are satisfied, the Bianchi identities read:

$$\begin{cases} dF^X + f_{YZ}{}^X A^Y \wedge F^Z &= 0 \\ dF^M - T_{XN}{}^M A^X \wedge F^N &= m^{MN} H_N \\ dH_M + T_{XM}{}^N A^X \wedge H_N &= d_{XMN} F^N \wedge F^X. \end{cases} \quad (2.12)$$

To see how the Higgs mechanism works in this more general case, let us give the gauge and tensor-gauge transformations of the fields (including the non-abelian transformations belonging to  $G_0$ , with parameter  $\epsilon^X$ ). One obtains:

$$\begin{cases} \delta A^X &= d\epsilon^X + f_{YZ}{}^X A^Y \epsilon^Z \equiv D\epsilon^X \\ \delta A^M &= d\Theta^M - T_{XN}{}^M A^X \Theta^N + T_{XN}{}^M A^N \epsilon^X - m^{MN} \Lambda_N \\ &\equiv D\Theta^M + T_{XN}{}^M A^N \epsilon^X - m^{MN} \Lambda_N \\ \delta B_M &= d\Lambda_M + T_{XM}{}^N A^X \wedge \Lambda_N - d_{XNM} F^X \Theta^N - T_{XM}{}^N B_N \epsilon^X \\ &\equiv D\Lambda_M - d_{XNM} F^X \Theta^N - T_{XM}{}^N B_N \epsilon^X, \end{cases} \quad (2.13)$$

with:

$$\begin{cases} \delta F^X &= f_{YZ}{}^X F^Y \epsilon^Z \\ \delta F^M &= T_{XN}{}^M F^N \epsilon^X \\ \delta H_M &= -T_{XM}{}^N H_N \epsilon^X. \end{cases} \quad (2.14)$$

Fixing the gauge of the tensor-gauge transformation as:

$$\begin{cases} A^M &\rightarrow A'^M = -m^{MN} \bar{\Lambda}_N \\ B_M &\rightarrow B'_M = B_M + D\bar{\Lambda}_M, \end{cases} \quad (2.15)$$

we find:

$$\begin{cases} F'^M &= m^{MN} B_N \\ H'_M &= DB_M \end{cases} \quad (2.16)$$

When the tensor-gauge is fixed as in (2.15), (2.16), the vectors  $A^M$  disappear from the spectrum while the tensors  $B_M$  acquire a mass. As anticipated in the introduction, this is in particular the starting point of the formulation adopted in the literature to describe  $D = 5$ ,  $N = 2$  supergravity coupled to massive tensor multiplets [15, 16, 17, 18, 19, 20].

However, let us observe that in this more general case the abelian gauge vectors  $A^M$ , providing the degrees of freedom needed to give a mass to the tensors via the anti-Higgs mechanism, are charged under the gauge algebra

$G_0$ . It is not possible to make the gauge transformation of the vectors  $A^M$  compatible with that of the  $A^X$  unless all together the vectors  $\{A^X, A^M\} \equiv A^\Lambda$  form the co-adjoint representation of the larger non semisimple gauge algebra  $G = G_0 \ltimes \mathbb{R}^{n_T}$ .

The relations so far obtained may then be written with the collective index  $\Lambda = (X, M)$ , in terms of structure constants  $f_{\Sigma\Gamma}^\Lambda$  restricted to the following non vanishing entries:

$$f_{\Sigma\Gamma}^\Lambda = (f_{XY}^Z, f_{XM}^N \equiv -T_{XM}^N), \quad (2.17)$$

and of the couplings:

$$m^{\Lambda M} \equiv \delta_N^\Lambda m^{NM}, \quad d_{\Lambda\Sigma M} \equiv \delta_\Lambda^X \delta_\Sigma^N d_{XNM}. \quad (2.18)$$

When this notation is used the FDA (2.5) reads:

$$\begin{cases} F^\Lambda & \equiv dA^\Lambda + \frac{1}{2}f_{\Sigma\Gamma}^\Lambda A^\Sigma \wedge A^\Gamma + m^{\Lambda M} B_M \\ H_M & \equiv dB_M + T_{\Lambda M}^N A^\Lambda \wedge B_N + d_{\Lambda\Sigma M} F^\Lambda \wedge A^\Sigma \end{cases} \quad (2.19)$$

with Bianchi identities:

$$\begin{cases} dF^\Lambda + (f_{\Sigma\Gamma}^\Lambda + m^{\Lambda M} d_{\Gamma\Sigma M}) A^\Sigma \wedge F^\Gamma = m^{\Lambda M} H_M \\ dH_M + (T_{\Lambda M}^N + m^{\Sigma N} d_{\Sigma\Lambda M}) A^\Lambda \wedge H_N = d_{\Lambda\Sigma M} F^\Lambda \wedge F^\Sigma \end{cases} \quad (2.20)$$

provided the following relations hold:

$$\begin{aligned} f_{[\Lambda\Sigma}^\Delta f_{\Gamma]\Delta}^\Pi &= 0 \\ [T_\Lambda, T_\Sigma] &= f_{\Lambda\Sigma}^\Gamma T_\Gamma \\ T_{\Lambda M}^{(N} m^{\Lambda|P)} &= 0 \\ m^{\Lambda N} T_{\Sigma N}^M &= f_{\Sigma\Gamma}^\Lambda m^{\Gamma M} \\ T_{\Lambda M}^N &= d_{\Lambda\Sigma M} m^{\Sigma N} \\ T_{[\Lambda|M}^N d_{\Gamma|\Sigma]N} &- (f_{[\Lambda|\Gamma}^\Delta + m^{\Delta N} d_{\Gamma[\Lambda N}) d_{\Delta|\Sigma]M} - \frac{1}{2} f_{\Lambda\Sigma}^\Delta d_{\Gamma\Delta M} = 0. \end{aligned} \quad (2.21)$$

Subject to the constraints (2.21), the system is covariant under the gauge transformations:

$$\begin{cases} \delta A^\Lambda &= d\epsilon^\Lambda + f_{\Sigma\Gamma}^\Lambda A^\Sigma \epsilon^\Gamma - m^{\Lambda M} \Lambda_M \\ \delta B_M &= d\Lambda_M + T_{\Lambda M}^N A^\Lambda \wedge \Lambda_N - d_{\Lambda\Sigma M} F^\Lambda \epsilon^\Sigma - T_{\Lambda M}^N \epsilon^\Lambda B_N \end{cases} \quad (2.22)$$

implying the gauge transformation of the field strengths:

$$\begin{cases} \delta F^\Lambda &= - (f_{\Sigma\Gamma}^\Lambda + m^{\Lambda M} d_{\Gamma\Sigma M}) \epsilon^\Sigma F^\Gamma \\ \delta H_M &= - (T_{\Lambda M}^N + m^{\Sigma N} d_{\Sigma\Lambda M}) \epsilon^\Lambda H_N \end{cases} \quad (2.23)$$

### 2.1.3 Extending the FDA with a more general gauge group

We now observe that the restrictions on the couplings (2.17) and (2.18) have been set to exactly reproduce eqs. (2.5) while exhibiting the fact that  $A^\Lambda$  collectively belong to the adjoint of some algebra  $G \supset G_0$ . Actually eqs. (2.5) and (2.21) allow in fact a more general gauge structure than the one declared in (2.17), (2.18). Let  $T_\Lambda \in \text{Adj } G$  be the gauge generators dual to  $A^\Lambda$ . For the case of (2.17),  $G$  has the non-semisimple structure  $G = G_0 \ltimes \mathbb{R}^{n_T}$ , and the generators  $T_X \in G_0$  may be realized in a block-diagonal way (with entries  $T_{XY}^Z = f_{XY}^Z$ ,  $T_{XM}^N = -f_{XM}^N$ ) while the  $T_M$  are off-diagonal (with entries  $T_{MX}^N = f_{XM}^N$ ). However, any gauge algebra  $G$  with structure constants  $f_{\Lambda\Sigma}^\Gamma$  may in principle be considered, provided it satisfies the constraints (2.21). In the general case, to match (2.21) one must also relax the restrictions on the couplings (2.17), (2.18) and allow for more general  $f_{\Lambda\Sigma}^\Gamma$  and  $d_{\Lambda\Sigma M}$ . This includes in particular the case

$$f_{XY}^M \neq 0, \quad d_{XYM} \neq 0 \quad (2.24)$$

which was considered in [20] and [25]. In this case,  $G$  has not anymore in general a non-semisimple structure, and  $G_0$  is not a subalgebra of  $G$ <sup>6</sup>. This implies that the vectors  $A^M$  do not decouple anymore at the level of gauge algebra, and this, at first sight, would be an obstruction to implement the Higgs mechanism on the 2-forms. However, this apparent obstruction may be simply overcome in the FDA framework, due to the freedom of redefining the tensor fields as [26]:

$$B_M \rightarrow B_M + k_{\Lambda\Sigma M} A^\Lambda \wedge A^\Sigma, \quad (2.25)$$

for any  $k_{\Lambda\Sigma M}$  antisymmetric in  $\Lambda, \Sigma$ . It is then possible to implement the Higgs mechanism with the tensor-gauge fixing (which includes a field redefinition as in (2.25)):

$$\begin{cases} A^X & \rightarrow A'^X = A^X \\ A^M & \rightarrow A'^M = -m^{MN} \bar{\Lambda}_N \\ B_M & \rightarrow B'_M = B_M - \frac{1}{2} d_{XYM} A^X \wedge A^Y + D \bar{\Lambda}_M \end{cases} \quad (2.26)$$

This still gives:

$$\begin{cases} F'^X & = F^X \\ F'^M & = m^{MN} B_N \\ H'_M & = D B_M \end{cases} \quad (2.27)$$

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<sup>6</sup>We acknowledge an enlightening discussion with Maria A. Lledó on this point.

provided that:

$$m^{MN}d_{[XY]N} = f_{XY}{}^M. \quad (2.28)$$

With this observation, we may now analyze in full generality which non trivial structure constants may be turned on in (2.19) in a way compatible with the anti-Higgs mechanism.

First of all, it is immediate to see that one must require:

$$f_{\Lambda M}{}^X = 0, \quad (2.29)$$

otherwise it is impossible to implement the Higgs mechanism. Indeed, such structure constants introduce a coupling to the gauge vectors  $A^M$  in the field-strengths  $F^X$  which is not possible to reabsorb by any field-redefinition.

Considering then the case:

$$f_{MN}{}^P \neq 0, \quad d_{MNP} \neq 0. \quad (2.30)$$

we see that  $f_{MN}{}^P$  would introduce a non-abelian interactions among the vectors  $A^M$  and in particular, for the case  $X = 0$ , this would imply that the  $A^M$  close a non-abelian gauge algebra. This case may be treated in a way quite similar to the case (2.24), since again we may use the freedom in (2.25) to absorb the non-abelian contribution to  $F^M$  in a redefinition of  $B_M$ . The anti-Higgs mechanism may then be implemented via the tensor-gauge fixing:

$$\begin{cases} A^X & \rightarrow A'^X = A^X \\ A^M & \rightarrow A'^M = -m^{MN}\bar{\Lambda}_N \\ B_M & \rightarrow B'_M = B_M - \frac{1}{2}d_{NPM}A^N \wedge A^P + D\bar{\Lambda}_M \end{cases} \quad (2.31)$$

giving, as before:

$$\begin{cases} F'^X & = F^X \\ F'^M & = m^{MN}B_N \\ H'_M & = DB_M \end{cases} \quad (2.32)$$

provided that:

$$m^{MQ}d_{[NP]Q} = f_{NP}{}^M. \quad (2.33)$$

This shows that also non-abelian gauge vectors  $A^M$  may be considered, and still may decouple from the gauge-fixed theory by giving mass to the tensors  $B_M$ . For this case, however, the constraints (2.21), together with (2.33), give the following conditions on the couplings:

$$\begin{cases} d_{MNP} & = d_{[MNP]} \\ m^{MN} & = +m^{NM} \end{cases} \quad (2.34)$$

Note that for the  $D = 5$ ,  $N = 2$  theory the matrix  $m^{MN}$  turns out to be antisymmetric, and this then implies, for that theory,  $f_{MN}{}^P = 0$ <sup>7</sup>.

## 2.2 Some observations on the properties of the FDA

A further observation concerns eqs. (2.20) and (2.23). In these equations, as in all the relations involving the physical field strengths  $F^\Lambda$  and  $H_M$ , the following objects appear:

$$\begin{aligned}\hat{f}_{\Sigma\Gamma}{}^\Lambda &\equiv f_{\Sigma\Gamma}{}^\Lambda + m^{\Lambda M} d_{\Gamma\Sigma M}; \\ \hat{T}_{\Lambda M}{}^N &\equiv T_{\Lambda M}{}^N + m^{\Sigma N} d_{\Sigma\Lambda M} = 2d_{(\Lambda\Sigma)M} m^{\Sigma N}.\end{aligned}\tag{2.35}$$

The generalized couplings  $\hat{f}_{\Sigma\Gamma}{}^\Lambda$  belong to a representation of the gauge algebra  $G$  which is not the adjoint, since they are not antisymmetric in the lower indices. In particular we find :

$$\begin{aligned}\hat{f}_{\Lambda\Sigma}{}^\Gamma m^{\Sigma M} &= \hat{T}_{\Lambda N}{}^M m^{\Sigma N} \\ \hat{f}_{\Lambda\Sigma}{}^\Gamma m^{\Lambda M} &= 0.\end{aligned}\tag{2.36}$$

However, the  $\hat{f}_{\Sigma\Gamma}{}^\Lambda$  and  $\hat{T}_{\Lambda N}{}^M$  can be understood as representations of generators  $\hat{f}_\Lambda$  and  $\hat{T}_\Lambda$  that still generate the gauge algebra  $G$ . Indeed the following relations hold (subject to the constraints (2.21)):

$$\begin{aligned}\left[\hat{f}_\Lambda, \hat{f}_\Sigma\right] &= -f_{\Lambda\Sigma}{}^\Gamma \hat{f}_\Gamma, \\ \left[\hat{T}_\Lambda, \hat{T}_\Sigma\right] &= f_{\Lambda\Sigma}{}^\Gamma \hat{T}_\Gamma.\end{aligned}\tag{2.37}$$

The generalized couplings  $\hat{f}$  and  $\hat{T}$  express the deformation of the gauge structure due to the presence of the tensor fields. In particular, only the structure constants of  $G_0$  are unchanged, corresponding to the fact that this is the algebra realized exactly in the interacting theory (2.19) after the anti-Higgs mechanism has taken place. The rest of the gauge algebra  $G$  is instead spontaneously broken by the anti-Higgs mechanism (which requires, if  $f_{XY}{}^M \neq 0$ , also a tensor redefinition, as explained in (2.26)). However, the entire algebra  $G$  is still realized, even if in a more subtle way, as eqs. (2.37)

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<sup>7</sup>In this case, since any invertible antisymmetric matrix may be chosen as symplectic metric, then equation  $m^{MN} T_{\Sigma N}{}^P = -T_{\Sigma N}{}^M m^{NP}$  (from (2.21)) implies that the generators  $T_{\Lambda M}{}^N$  belong to a symplectic representation of the gauge group [9, 10, 11].

show. From a physical point of view, this is expected by a counting of degrees of freedom, since the degrees of freedom required to make a two-index tensor massive are the ones of a gauge vector connection <sup>8</sup>, so that also the vectors  $A^M$ , besides the  $A^X$ , are expected to be massless gauge vectors. This algebra indeed closes provided the Jacobi identities  $f_{[\Lambda\Sigma}{}^\Delta f_{\Gamma\Delta]}{}^\Pi = 0$  are satisfied. We find indeed, using (2.37):

$$\begin{aligned} \left[ \left[ \hat{f}_{[\Lambda}, \hat{f}_{\Sigma]} \right], \hat{f}_{\Gamma} \right]_{\Delta}{}^{\Theta} &= -f_{[\Lambda\Sigma}{}^{\Omega} f_{\Gamma]\Omega}{}^{\Pi} \hat{f}_{\Pi\Delta}{}^{\Theta} = 0 \\ \left[ \left[ \hat{T}_{[\Lambda}, \hat{T}_{\Sigma]} \right], \hat{T}_{\Gamma} \right]_M{}^N &= -f_{[\Lambda\Sigma}{}^{\Pi} f_{\Gamma]\Pi}{}^{\Delta} \hat{T}_{\Delta M}{}^N = 0 \end{aligned} \quad (2.38)$$

The hatted generators  $\hat{f}$ ,  $\hat{T}$  play the role of *physical couplings* when the gauge structure is extended to include charged tensors. They have then to be considered as the appropriate generators of the free differential structure. It may be useful to recast the theory in terms of all the couplings appearing in the Bianchi identities (2.20), that is the hatted generators and the symmetric part  $d_{(\Lambda\Sigma)M}$  of the Chern–Simons-like coupling  $d_{\Lambda\Sigma M}$ . This is done by the field redefinition:

$$B_M \rightarrow \tilde{B}_M = B_M + \frac{1}{2} d_{[\Lambda\Sigma]M} A^\Lambda \wedge A^\Sigma \quad (2.39)$$

so that the FDA takes the form:

$$\begin{cases} F^\Lambda &\equiv dA^\Lambda + \frac{1}{2} \hat{f}_{\Sigma\Gamma}{}^\Lambda A^\Sigma \wedge A^\Gamma + m^{\Lambda M} \tilde{B}_M \\ H_M &\equiv d\tilde{B}_M + \frac{1}{2} \hat{T}_{\Lambda M}{}^N A^\Lambda \wedge \tilde{B}_N + d_{(\Lambda\Sigma)M} F^\Lambda \wedge A^\Sigma + \\ &\quad + \mathcal{K}_{M\Lambda\Sigma\Gamma} A^\Lambda \wedge A^\Sigma \wedge A^\Gamma \end{cases} \quad (2.40)$$

and the constraints (2.21) in the new formulation read, after introducing  $\tilde{f}_{\Lambda\Sigma}{}^\Gamma \equiv \hat{f}_{[\Lambda\Sigma]}{}^\Gamma$ :

$$\begin{aligned} +\tilde{f}_{[\Lambda\Sigma}{}^\Pi \tilde{f}_{\Gamma]\Pi}{}^\Delta &= 2m^{\Delta M} \mathcal{K}_{M[\Lambda\Sigma\Gamma]} \\ \hat{T}_{[\Lambda M}{}^N \hat{T}_{\Sigma]N}{}^P &= \tilde{f}_{\Lambda\Sigma}{}^\Gamma \hat{T}_\Gamma{}^P + 12\mathcal{K}_{M\Lambda\Sigma\Gamma} m^{\Gamma P} \\ \hat{T}_{\Lambda M}{}^N m^{\Lambda P} &= 0 \\ \frac{1}{2} m^{\Lambda N} \hat{T}_{\Sigma N}{}^M &= \tilde{f}_{\Sigma\Gamma}{}^\Lambda m^{\Gamma M} \\ \hat{T}_{\Lambda M}{}^N &= 2d_{(\Lambda\Sigma)M} m^{\Sigma N} \\ \hat{T}_{[\Lambda|M}{}^N d_{(\Sigma)\Gamma)N} &- 2\hat{f}_{[\Lambda|\Gamma}{}^\Delta d_{(\Sigma)\Delta)M} - \tilde{f}_{\Lambda\Sigma}{}^\Delta d_{(\Gamma\Delta)M} = -6\mathcal{K}_{M\Lambda\Sigma\Gamma} \\ \mathcal{K}_{N[\Sigma\Gamma\Delta} \hat{T}_{\Lambda]M}{}^N &- 3\mathcal{K}_{M\Theta[\Lambda\Sigma\Gamma\Delta]}{}^\Theta = 0. \end{aligned} \quad (2.41)$$

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<sup>8</sup>Indeed, the on-shell degrees of freedom of a massless (2-index) tensor and of a vector in  $D$  dimensions are  $(D-2)(D-3)/2$  and  $(D-2)$  respectively, while the ones of a massive tensor are  $(D-1)(D-2)/2 = (D-2)(D-3)/2 + (D-2)$ .

In eqs. (2.40) and (2.41) we have introduced the definition:

$$\mathcal{K}_{M\Lambda\Sigma\Gamma} = \frac{1}{2}\hat{f}_{[\Lambda\Sigma}{}^\Delta d_{(\Gamma]\Delta)M} + \frac{2}{3}d_{(\Delta[\Sigma)M}\hat{f}_{\Lambda]\Gamma}{}^\Delta. \quad (2.42)$$

that could also be found by directly studying the closure of the FDA (2.40) without referring to its derivation from (2.19).

Eq. (2.40), which is expressed in terms of the physical couplings only, is completely equivalent to (2.19). This is in fact the formulation used in [25], for the study of  $N = 8$  supergravity in 5 dimensions. However, as eqs. (2.41) show, in the formulation (2.40) the gauge structure is not completely manifest, because the Jacobi identities for the “structure constants”  $\hat{f}_{\Lambda\Sigma}{}^\Gamma$  fail to close.

Equation (2.19) (or, equivalently, (2.40)) is the most general FDA involving vectors and 2-index antisymmetric tensors. Any other possible deformation of (2.19) is indeed trivial.

As a final remark, let us observe that, given the definitions (2.19), the FDA still enjoys a scale invariance under the transformation, with parameter  $\alpha$ :

$$\begin{cases} m^{MN} & \rightarrow \alpha m^{MN} \\ B_M & \rightarrow \frac{1}{\alpha} B_M \\ d_{\Lambda\Sigma M} & \rightarrow \frac{1}{\alpha} d_{\Lambda\Sigma M} \end{cases} \quad (2.43)$$

Such invariance is useful for fixing the overall normalization of the 2-form contributions to the Chern–Simons terms when constructing the Lagrangian.

### 3 3-form potentials

In this section we consider the extension of the bosonic FDA in the presence of  $p$ -form potentials with  $p \geq 3$ . We focus our attention in particular to 3-forms. Then, in subsection 3.1 we outline a simple generalization of the bosonic FDA and of the related Higgs mechanism providing a mass to higher degree  $p$ -form potentials.

The inclusion of 3-forms in a FDA has already been considered in ref. [27] where its role in gauged maximal supergravity was analyzed, with particular attention to the representation of the duality group to which they belong.

Here, we consider the 3-forms as propagating fields with their gauge-invariant field-strengths (4-form curvatures) appropriately defined, that is

defined in such a way that the Bianchi identities are satisfied in terms of gauge-invariant quantities. These requirements are satisfied provided the various invariant tensors entering the definition of the curvatures satisfy some constraints that generalize those already found in the absence of 3-forms (see eqs. (2.21) in section 2).

Note that if we think of  $p$ -forms as coming from the dualization of the scalar fields of some supergravity theory we can obtain new theories where a subset of the original scalars are dualized into  $p$ -forms. For example, if we consider  $D = 5$ ,  $N = 2$  standard supergravity, some scalar fields in the hypermultiplet sector could be dualized into 3-forms, generating new couplings in the theory. By dimensional reduction to  $D = 4$  of that theory, one should obtain the  $D = 4$ ,  $N = 2$  supergravity coupled to scalar-tensor multiplets of ref. [4, 5]. More general examples should be worked out in the appropriate framework.

Since in our case we are interested in the general structure of the bosonic FDA, the 3-form will be given a generic representation (lower) index  $\alpha$  and we shall denote it as  $S_\alpha$ . To construct the new FDA we add  $S_\alpha$  to the r.h.s. of the 3-form field strength  $H_M$  and try to define a new 4-form field strength,  $G_\alpha$ , in such a way that the Bianchi identities close. Let us define the following FDA:

$$F^\Lambda = dA^\Lambda + \frac{1}{2} f_{\Sigma\Gamma}{}^\Lambda A^\Sigma A^\Gamma + m^{\Lambda M} B_M \quad (3.1)$$

$$H_M = dB_M + T_{\Lambda M}{}^N A^\Lambda B_N + d_{\Lambda\Sigma M} F^\Lambda A^\Sigma + L_{\Lambda\Sigma\Gamma M} A^\Lambda A^\Sigma A^\Gamma + k_M^\alpha S_\alpha \quad (3.2)$$

$$\begin{aligned} G_\alpha = & dS_\alpha + T_{\Lambda\alpha}{}^\beta A^\Lambda S_\beta + T_{\alpha\Lambda}{}^M F^\Lambda B_M + T'_{\alpha\Lambda}{}^M A^\Lambda H_M + \\ & + L_{\Lambda\Sigma\Gamma\alpha} A^\Lambda A^\Sigma F^\Gamma + M_{\Lambda\Sigma\Gamma\Delta\alpha} A^\Lambda A^\Sigma A^\Gamma A^\Delta + N_{\alpha\Lambda\Sigma}{}^M A^\Lambda A^\Sigma B_M + \\ & + N_\alpha{}^{MN} B_M B_N + R_{\alpha\Lambda\Sigma} F^\Lambda F^\Sigma \end{aligned} \quad (3.3)$$

Here  $k_M^\alpha$  is a matrix intertwining between the  $M$ - and the  $\alpha$ - representations, and all the tensors appearing as coefficients of the wedge products of forms in eqs. (3.3) are constant tensors in the representations defined by the structure of their indices, invariant under the action of the group  $G$ .

Differentiations of the curvatures  $F^\Lambda$ ,  $H_M$ ,  $G_\alpha$  gives the following Bianchi identities which are explicitly covariant under all the local symmetries:

$$DF^\Lambda = dF^\Lambda + \hat{f}_{\Sigma\Gamma}{}^\Lambda A^\Sigma F^\Gamma = m^{\Lambda M} H_M \quad (3.4)$$

$$DH_M = dH_M + T_{\Lambda M}^{(H)N} A^\Lambda H_N = (d_{(\Lambda\Sigma)M} + k_M^\alpha R_{\alpha\Lambda\Sigma}) F^\Lambda F^\Sigma + k_M^\alpha G_\alpha \quad (3.5)$$

$$DG_\alpha = dG_\alpha + T_{\Lambda\alpha}^{(G)\beta} A^\Lambda G_\beta = (T_{\Lambda\alpha}{}^M + T'_{\Lambda\alpha}{}^M + 2R_{\alpha\Lambda\Sigma} m^{\Sigma M}) F^\Lambda H_M, \quad (3.6)$$



where we have defined:

$$\hat{f}_{\Sigma\Gamma}{}^\Lambda = f_{\Sigma\Gamma}{}^\Lambda + m^{\Lambda M} d_{\Gamma\Sigma M} \quad (3.7)$$

$$T_{\Lambda M}^{(H)N} = T_{\Lambda M}{}^N + d_{\Sigma\Lambda M} m^{\Sigma N} + k_M^\alpha T'_{\Lambda\alpha}{}^N \quad (3.8)$$

$$T_{\Lambda\alpha}^{(G)\beta} = T_{\Lambda\alpha}{}^\beta + T'_{\Lambda\alpha}{}^M k_M^\beta \quad (3.9)$$

provided that a set of relations among the couplings are satisfied. This is a complicated system of equations which may be significantly simplified if we consider particular physical situations. Since we will not develop further the general case here, we list the set of constraints in Appendix A. Among these identities, we just analyze the first equation in the list, namely:

$$m^{\Lambda M} k_M^\alpha = 0.$$

It is interesting to discuss the physical meaning of such constraint. Roughly speaking, it expresses the fact that, if we introduce in  $H_M$  a non trivial coupling  $k_M^\alpha$  to give mass to the 3-form  $S_\alpha$ , then the mass term  $m^{\Lambda M}$  for the corresponding 2-form  $B_M$  must be zero. In more detail, restricting our analysis to the case of  $k_M^\alpha$  with maximal rank, we will distinguish the possible different situations which can arise depending on the presence or not of the coupling to the 3-form potential,  $k_M^\alpha$ . We have then two substantially different cases, corresponding to the range of the index  $\alpha$  being greater or lower to the range of  $M$  (shortly  $|\alpha| \geq |M|$  or  $|\alpha| < |M|$ ).

In the first case ( $|\alpha| \geq |M|$ ), all the 3-form potentials may become massive by absorbing the degrees of freedom of the 3-form field-strengths  $H_M$ ; this, however, can only be possible if the d.o.f. of the  $B_M$  potentials are those of massless fields, so that the Higgs mechanism giving mass to  $B_M$  is actually forbidden. Let us define, for this case,  $S_M \equiv S_\alpha k_M^\alpha$ , and call  $S_a$  the residual 3-form potentials (so that  $S_\alpha \rightarrow (S_M, S_a)$ ). Then, by a proper redefinition of the fields  $S_\alpha$  and  $B_M$ , the FDA can always be rewritten in the following form

$$F^\Lambda = dA^\Lambda + \frac{1}{2} f_{\Sigma\Gamma}{}^\Lambda A^\Sigma A^\Gamma \quad (3.10)$$

$$H_M = dB_M + T_{\Lambda M}{}^N A^\Lambda B_N + S_M \quad (3.11)$$

$$G_M = dS_M + T_{\Lambda M}{}^N A^\Lambda S_N + T_{\Lambda M}{}^N F^\Lambda B_N + R_{\Lambda\Sigma M} F^\Lambda F^\Sigma \quad (3.12)$$

$$G_a = dS_a \quad (3.13)$$

with Bianchi identities:

$$DF^\Lambda = dF^\Lambda + f_{\Sigma\Gamma}^\Lambda A^\Sigma F^\Gamma = 0 \quad (3.14)$$

$$DH_M = dH_M + T_{\Lambda M}^N A^\Lambda H_N = (G_M - R_{\Lambda\Sigma M} F^\Lambda F^\Sigma) \quad (3.15)$$

$$DG_M = dG_M + T_{\Lambda M}^N A^\Lambda G_N = T_{\Lambda M}^N F^\Lambda H_N \quad (3.16)$$

$$dG_a = 0, \quad (3.17)$$

where

$$T_{\Lambda M}^N = k_M^\alpha T_{\Lambda\alpha}^N \quad (3.18)$$

$$T_{\Lambda\alpha}^\beta = T_{\Lambda\alpha}^M k_M^\beta. \quad (3.19)$$

In the present case the set of constraints of Appendix A reduce dramatically to the following simple set of constraints among the couplings :

$$f_{[\Lambda\Sigma}^\Pi f_{\Gamma]\Pi}^\Delta = 0 \quad (3.20)$$

$$T_{[\Lambda|M}^P T_{\Sigma]P}^N = \frac{1}{2} f_{\Lambda\Sigma}^\Gamma T_{\Gamma M}^N \quad (3.21)$$

$$T_{\Lambda M}^N k_N^\alpha = k_N^\beta T_{\Lambda\beta}^\alpha \quad (3.22)$$

$$T_{(\Lambda|M}^N R_{\Sigma)\Gamma N} - 2f_{\Lambda(\Sigma}^\Delta R_{\Gamma)\Delta M} = 0 \quad (3.23)$$

One can verify that a further differentiation of the Bianchi identities (3.14) gives identically zero.

On the other hand, if  $|\alpha| < |M|$ , so that  $M = (\alpha, \tilde{M})$ , a subset of 2-index potentials,  $B_{\tilde{M}}$ , does not disappear from the spectrum. In this case the usual Higgs mechanism discussed in section 2 may take place for the 2-forms  $B_{\tilde{M}}$ , so that all the relations involving 2- and 3-form field-strengths are still valid; however, now the explicit solution of the constraints (A.1) - (A.20) is more involved, and we leave it to forthcoming work, where the appropriate applications to five dimensional supergravity will be discussed.

Note that, in ref. [27], the 3-form potentials were introduced on the r.h.s. of  $H_M$  in the form  $\hat{T}_{\Lambda M}^N S_N^\Lambda$ , where  $\Lambda$  is an index in the duality group of a supergravity theory. We have seen that if we want the 3-form potentials as propagating physical fields in a supergravity Lagrangian where the scalar fields of some supermultiplet have been dualized, then  $\hat{T}_{\Lambda M}^N$  must be substituted with  $T_{\Lambda M}^N$ , since  $m^{\Lambda M} = 0$ .

The construction of a supergravity theory of this kind, which amounts to extend the FDA to a super-FDA containing the gravitino spinor 1-forms,

particularly in the case of an  $N = 2$  five dimensional supergravity, will be left to a future investigation. However, we may observe that there is no obstruction in enlarging the FDA to a super-FDA, at least in five dimensions. Indeed, one can modify the definition of the curvatures in superspace the gravitino 1-forms  $\psi_A$ , in such a way that the closure of the FDA is still achieved. Let us consider, as an example, the case where some of the scalars in the hypermultiplet sector have been dualized into  $S_\alpha$ . As far as the 3-forms in superspace are concerned, the proper definition of their curvatures is the following:

$$G_\alpha = dS_\alpha + T_{\Lambda\alpha}{}^\beta A^\Lambda S_\beta + T_{\Lambda\alpha}{}^M (F^\Lambda - X^\Lambda \bar{\psi}_A \psi^A) B_M + \\ + L_{\Lambda\Sigma\alpha} (F^\Lambda - X^\Lambda \bar{\psi}_A \psi^A) (F^\Sigma - X^\Sigma \bar{\psi}_A \psi^A) + \omega_\alpha^{AB} \bar{\psi}_A \Gamma_{ab} \psi_B V^a V^b. \quad (3.24)$$

One immediately sees that the super-FDA at zero curvatures (minimal FDA) is consistent owing to the five dimensional Fierz identities

$$\bar{\psi}_C \Gamma^{ab} \psi_A \bar{\psi}_B \Gamma_b \psi^B = 0, \quad (3.25)$$

provided  $\nabla \omega_\alpha^{AB} = 0$ . Analogously to the case analyzed in [4],  $\omega_\alpha^{AB}$  has an interpretation as the  $SU(2)$  connection whose coordinate index is Hodge-dualized into a 3-form tensor index  $\alpha$ . In fact we expect that the dimensional reduction of this theory would reproduce the  $D = 4$   $N = 2$  supergravity coupled to scalar-tensor multiplets given in [4, 5].

### 3.1 A possible generalization to $p$ -form potentials

The analysis of the previous sections admits a generalization allowing the presence of  $p$ -form potentials, with  $p > 3$ . Indeed, a possible generalization of the Higgs mechanism, generating massive  $p$ -forms, can be achieved

introducing the following set of curvatures:

$$F^\Lambda = dA^\Lambda + \frac{1}{2}f_{\Sigma\Gamma}{}^\Lambda A^\Sigma A^\Gamma \quad (3.26)$$

$$F^M = dA^M - T_{\Lambda N}{}^M A^\Lambda A^N + m^{MN} B_N \quad (3.27)$$

$$H_M^{(3)} = dB_M^{(2)} + T_{\Lambda M}{}^N A^\Lambda B_N^{(2)} + d_{\Lambda N M} F^\Lambda A^N \quad (3.28)$$

$$H_\alpha^{(3)} = dB_\alpha^{(2)} + T_{\Lambda\alpha}{}^\beta A^\Lambda B_\beta^{(2)} + S_\alpha^{(3)} \quad (3.29)$$

$$G_\alpha^{(4)} = dS_\alpha^{(3)} + T_{\Lambda\alpha}{}^\beta A^\Lambda S_\beta^{(3)} + T_{\Lambda\alpha}{}^\beta F^\Lambda B_\beta^{(2)} \quad (3.30)$$

$$\dots = \dots\dots\dots$$

$$H_\alpha^{(n-1)} = dB_\alpha^{(n-2)} + T_{\Lambda\alpha}{}^\beta A^\Lambda B_\beta^{(n-2)} + S_\alpha^{(n-1)} \quad (3.31)$$

$$G_\alpha^{(n)} = dS_\alpha^{(n-1)} + T_{\Lambda\alpha}{}^\beta A^\Lambda S_\beta^{(n-1)} + T_{\Lambda\alpha}{}^\beta F^\Lambda B_\beta^{(n-2)} \quad (3.32)$$

Differentiating the curvatures, we get the following set of covariant Bianchi identities:

$$DF^\Lambda = dF^\Lambda + f_{\Sigma\Gamma}{}^\Lambda A^\Sigma F^\Gamma = 0 \quad (3.33)$$

$$DF^M = dF^M - T_{\Lambda N}{}^M A^\Lambda F^N = m^{MN} H_N \quad (3.34)$$

$$DH_M = dH_M + T_{\Lambda M}{}^N A^\Lambda H_N = d_{\Lambda N M} F^\Lambda F^N \quad (3.35)$$

$$DH_\alpha = dH_\alpha + T_{\Lambda\alpha}{}^\beta A^\Lambda H_\beta = G_\alpha \quad (3.36)$$

$$DG_\alpha = dG_\alpha + T_{\Lambda\alpha}{}^\beta A^\Lambda G_\beta = T_{\Lambda\alpha}{}^\beta F^\Lambda H_\beta \quad (3.37)$$

$$\dots = \dots\dots\dots$$

$$DH_\alpha^{(n-1)} = dH_\alpha^{(n-1)} + T_{\Lambda\alpha}{}^\beta A^\Lambda H_\beta^{(n-1)} = G_\alpha^{(n)} \quad (3.38)$$

$$DG_\alpha^{(n)} = dG_\alpha^{(n)} + T_{\Lambda\alpha}{}^\beta A^\Lambda G_\beta^{(n)} = T_{\Lambda\alpha}{}^\beta F^\Lambda H_\beta^{(n-1)} \quad (3.39)$$

provided that the following relations hold:

$$f_{[\Lambda\Sigma}{}^\Delta f_{\Gamma]\Delta}{}^\Pi = 0 \quad (3.40)$$

$$[T_\Lambda, T_\Sigma] = f_{\Lambda\Sigma}{}^\Gamma T_\Gamma \quad (3.41)$$

$$T_{\Lambda P}{}^M m^{PN} + m^{MP} T_{\Lambda P}{}^N = 0 \quad (3.42)$$

$$T_{\Lambda N}{}^M = -d_{\Lambda N P} m^{MP} \quad (3.43)$$

$$T_{\Lambda N}{}^M = d_{\Lambda P N} m^{PM} \quad (3.44)$$

$$T_{\Lambda M}{}^P d_{\Sigma N P} + d_{\Sigma P M} T_{\Lambda N}{}^P = f_{\Lambda\Sigma}{}^\Gamma d_{\Gamma N M} . \quad (3.45)$$

## 4 $N = 2$ , $D = 4$ Supergravity

Our most important application of the previous formalism, restricted to 2-form potentials (see Section 2), is the explicit construction of the  $N = 2$ ,  $D = 4$  supergravity theory coupled to vector-tensor multiplets, that is those multiplets that can be obtained from vector multiplets by Hodge-dualization of, say, the imaginary part of a subset of the complex scalar fields parametrizing the special manifold. At our knowledge, as anticipated in the introduction, the construction of such theory in full generality has not been achieved so far, even if important steps in that direction have been given in ref. [21], where the four dimensional theory was obtained by dimensional reduction from five dimensions and the ensuing properties thoroughly analyzed. However, this approach does not catch the most general theory, being restricted to models with a five dimensional uplift. A general approach containing vector-tensor multiplets has been developed in ref. [23] by use the framework of the embedding tensor, but it is restricted to supersymmetric rigid gauge theories.

The general analysis we develop in this section relies on the solution of Bianchi identities in superspace which, besides giving the general supersymmetry transformation laws and the constraints on the geometry of the relevant  $\sigma$ -models, also allows us in principle to retrieve the equations of motion of the theory. It would be of course desirable to have the supersymmetric Lagrangian to put in evidence the couplings of the theory, but we leave this construction to a forthcoming paper. However, the knowledge of the explicit expression of the fermion shifts will allow us to reconstruct the scalar potential.

Let us consider  $N = 2$  supergravity in four dimensions with field content given by:

$$(V_\mu^a, \psi_{A\mu}, \psi_\mu^A, A_\mu^0),$$

(where  $a$  and  $\mu$  denote space-time indices respectively flat and curved,  $A = 1, 2$  is an  $SU(2)$  index and we have decomposed the gravitino in chiral ( $\psi_A$ ) and anti-chiral ( $\psi^A$ ) components), coupled to  $n_V$  vector multiplets:

$$(A_\mu, \lambda^A, z)^r, \quad r = 1, \dots, n_V,$$

where  $z^r$  are holomorphic coordinates on the special manifold  $\mathcal{M}_V$  spanned by its scalar sector and  $\lambda^{rA}$  are chiral spin-1/2 fields (with complex conjugate

antichiral component  $\lambda_A^{\bar{r}}$ ), and to  $n_T$  vector-tensor multiplets:

$$(B_{M\mu\nu}, A_\mu^M, \chi^{mA}, \phi^m), \quad M, m = 1, \dots, n_T,$$

where  $\phi^m$  are real coordinates on the real manifold  $\mathcal{M}_T$  spanned by its scalar sector ( $m$  is a coordinate index on  $\mathcal{M}_T$  while  $M$  is a representation index of the non-semisimple gauge group  $G$ ) and  $\chi^{mA}$  are Majorana spinors, not decomposed in chiral components. Written in this way, the tensor multiplets are naturally interpreted as obtained from  $n_T$  extra vector multiplets by Hodge-dualization of the imaginary part  $Y^m$  of the corresponding holomorphic scalars  $z^m$ , as will be apparent in the following (see eq. (4.20) and Appendix C).

Let us now study here the supersymmetric FDA of this theory. We shall let all the vectors  $A^\Lambda$  be the gauge vectors of a non abelian non-semisimple algebra  $G = G_0 \ltimes \mathbb{R}^M$  and the tensors  $B_M$  to be in a representation of  $G_0$ .<sup>9</sup> In the interacting theory, the anti-Higgs mechanism will take place so that the vectors  $A^M$  will provide the degrees of freedom to give mass to the tensors  $B_M$ . In this way the gauge algebra will be broken to its subalgebra  $G_0$  ( $\dim G_0 = n_V + 1$ ) spanned by the vectors  $(A^0, A^r)$ . It will then be useful to adopt a collective gauge-vector index  $\Lambda = (X, M) = 0, 1, \dots, n_V + n_T$  (with  $X = 0, 1, \dots, n_V$ ) running over all the vectors of the theory.

In order for the FDA to close at the supersymmetric level, it will be necessary to include among the defining bosonic fields of the tensor multiplet sector, besides the vectors  $A^M$  and the tensors  $B_M$ , also their Hodge duals  $A_M$  ( $*dA^M \propto m^{MN}dA_N$ ) and  $Y^m$ . The dual gauge vectors  $A_M$  (undergoing a dual Higgs mechanism, since they take mass by eating the degrees of freedom of the dualized scalars  $Y^m$ ) will indeed appear in the supercurvature of the tensor field-strengths  $H_M$ . If we would not include them, the Bianchi identities would show up inconsistencies. Then, since the fields  $Y^m$  have to be included for a correct description of the dynamics of the theory, it is convenient to adopt a complex notation also for the vector-tensor sector and work with holomorphic coordinates  $z^m \equiv \phi^m + iY^m$  together with their complex conjugates  $\bar{z}^{\bar{m}} \equiv \phi^m - iY^m$ . Correspondingly, we will generally decompose the Majorana spinor  $\chi^{mA}$  in chiral components denoted by  $\lambda^{mA}, \lambda_A^{\bar{m}}$ . Using this notation, it will be natural to introduce a collective holomorphic world-index  $i = (r, m) = 1, \dots, n_V + n_T$ , in parallel to what has been done for gauge

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<sup>9</sup>As discussed in section 2, even if we start from a more general  $G \supset G_0 \ltimes \mathbb{R}^{n_T}$ , we can always retrieve this case by a suitable redefinition of the 2-forms  $B_M$ .

indices. At this point we note that using the collective index formalism the theory looks much like the standard  $N = 2$  supergravity coupled to vector multiplets only, and this explains, as we will see in the following, that most of the results coming from Bianchi Identities will look formally like those of the standard  $N = 2$  supergravity. Since then the Free Differential Algebra involves both the antisymmetric tensors  $B_M$  and the scalars  $Y^m$ , we expect that the closure of the Bianchi identities would imply the duality relation between them. Actually, this is what happens implying that the dualization relation is valid only on-shell. As a consequence, the on-shell geometry will look quite different from its off-shell (Kähler–Hodge) counterpart. In particular, in absence of a factorization of the two  $\sigma$ -models, the off-shell Kähler–Hodge structure is completely destroyed since the metric is not even hermitian.

We further note that, exactly like the five dimensional case, for the  $N = 2$  four dimensional theory the massive vector-tensor multiplets are short, BPS multiplets.<sup>10</sup> They are therefore charged and this in turn requires for CPT invariance that the vector-tensor multiplet sector always includes an even number of tensor fields. We have chosen to take as vectors belonging to the vector-tensor multiplets the vectors  $A^M$  with an upper representation index, since these are the degrees of freedom participating to the anti-Higgs mechanism giving mass to the tensors  $B_M$ . As a byproduct, the Hodge-dual vectors  $A_M$  become massive by eating the degrees of freedom of the scalars  $Y^m$  (Hodge-dual to the tensors  $B_M$ ).

Passing from the bosonic FDA to the one for  $N = 2$  supergravity in four dimensions, the relations defining the algebra get modified in various directions. First of all one has to include the FDA of pure supergravity:

$$\mathcal{R}^a_b = d\omega^a_b - \omega^a_c \wedge \omega^c_b \quad (4.1)$$

$$\mathcal{T}^a = dV^a - \omega^a_b V^b - i\bar{\psi}_A \gamma^a \psi^A \quad (4.2)$$

$$\rho_A = d\psi_A - \frac{1}{4}\omega_{ab}\gamma^{ab}\psi_A + \frac{i}{2}\mathcal{Q}\psi_A \quad (4.3)$$

$$\rho^A = d\psi^A - \frac{1}{4}\omega_{ab}\gamma^{ab}\psi^A - \frac{i}{2}\mathcal{Q}\psi^A. \quad (4.4)$$

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<sup>10</sup>This has to be contrasted to what happens for the scalar-tensor multiplets where the tensor field is Hodge-dual to a scalar in the hypermultiplet sector. In that case, the multiplet becomes massive by introducing an appropriate coupling to a vector multiplet. In our case, instead, the fields  $A_M$  and  $Y^m$  do not have a spinor partner, but act as bosonic Lagrange multipliers in the theory.

where with  $\mathcal{Q} \equiv Q_r dz^r + Q_{\bar{r}} d\bar{z}^{\bar{r}}$  we denote the gauged  $U(1)$ -Kähler composite connection of special geometry.<sup>11</sup>

Secondly, the bosonic curvatures introduced in section 2 for the field-strengths have to be generalized to their supersymmetric extension, that is:

$$H_M = dB_M + T_{\Lambda M}^N A^\Lambda B_N + iP_M \bar{\psi}_A \gamma_a \psi^A V^a + \left( d_{\Lambda \Sigma M} A^\Sigma + \hat{T}_{\Lambda M}^N A_N \right) \left( F^\Lambda - L^\Lambda \bar{\psi}^A \psi^B \epsilon_{AB} - \bar{L}^\Lambda \bar{\psi}_A \psi_B \epsilon^{AB} \right) \quad (4.5)$$

$$F^\Lambda = dA^\Lambda + \frac{1}{2} f_{\Sigma\Gamma}^\Lambda A^\Sigma A^\Gamma + m^{\Lambda M} B_M + L^\Lambda \bar{\psi}^A \psi^B \epsilon_{AB} + \bar{L}^\Lambda \bar{\psi}_A \psi_B \epsilon^{AB} \quad (4.6)$$

$$F_M = dA_M + \hat{T}_{\Lambda M}^N A^\Lambda A_N + M_M \bar{\psi}^A \psi^B \epsilon_{AB} + \bar{M}_M \bar{\psi}_A \psi_B \epsilon^{AB} \quad (4.7)$$

Here  $P_M$  is a real section on the  $\sigma$ -model,  $L^\Lambda$  and  $\bar{L}^\Lambda$  are the sections of special geometry, while  $M_M$  and  $\bar{M}_M$  are new sections in the given representation of  $G_0$ .

Finally, the FDA has to be enlarged to include the 1-form gauged field-strengths for the 0-form scalars and spinors belonging to the representations of supersymmetry:

$$Dz^i = dz^i + k_\Lambda^i A^\Lambda - k^{iM} A_M \quad (4.8)$$

$$\nabla \lambda^{iA} = d\lambda^{iA} - \frac{1}{4} \omega_{ab} \gamma^{ab} \lambda^{iA} + \Gamma^i_j \lambda^{jA} \quad (4.9)$$

where  $k_\Lambda^i$  are the holomorphic part of Killing vectors in the adjoint representation of the algebra  $G$  while  $k^{iM} \equiv k^{mN} \delta_m^i$  are purely imaginary Killing vectors in the coadjoint representation of the invariant subgroup of  $G$ , satisfying  $k^{mM} = -k^{\bar{m}\bar{M}}$ . This choice corresponds to the requirement that the vectors  $A_M$  undergo the Higgs mechanism by eating the imaginary part  $Y^m$  of the scalars  $z^m$ . This will be consistent with the solution of the Bianchi identities.

The (on-shell) solution of the Bianchi identities for the value of the super-curvatures, up to three-fermion terms, is given in detail in Appendix B.1. In the solution new structures appear, which are defined by a set of constraints. On the scalar geometry one finds:

$$D_i L^\Lambda = f_i^\Lambda - i Q_m \delta_i^m L^\Lambda, \quad D_{\bar{i}} L^\Lambda = -i Q_{\bar{m}} \delta_{\bar{i}}^{\bar{m}} L^\Lambda \quad (4.10)$$

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<sup>11</sup>For its definition in terms of the ungauged one and for all the notation concerning special geometry, we refer the reader to the standard  $N = 2$ ,  $D = 4$  supergravity of ref. [28].



$$D_i M_M = g_{Mi} - i Q_m \delta_i^m M_M, \quad D_{\bar{i}} M_M = -i Q_{\bar{m}} \delta_{\bar{i}}^{\bar{m}} M_M \quad (4.11)$$

$$h_a = \frac{i}{2} (Q_m D_a z^m + Q_{\bar{m}} D_a \bar{z}^{\bar{m}}) \quad (4.12)$$

$$P_M = - \left[ 2d_{(\Lambda\Sigma)M} L^\Lambda \bar{L}^\Sigma + \hat{T}_{\Lambda M}^N \left( \bar{L}^\Lambda M_N + L^\Lambda \bar{M}_N \right) \right] \quad (4.13)$$

$$0 = d_{(\Lambda\Sigma)M} L^\Lambda L^\Sigma + \hat{T}_{\Lambda M}^N L^\Lambda M_N \quad (4.14)$$

$$0 = 2d_{(\Lambda\Sigma)M} L^\Lambda f_i^\Sigma + \hat{T}_{\Lambda M}^N (L^\Lambda g_{iN} + f_i^\Lambda M_N) \quad (4.15)$$

$$D_i P_M = 2h_{Mi} \quad (4.16)$$

The new quantities  $g_{Mi}$ ,  $h_{Mi}$  are defined by eqs. (4.11), (4.16). Moreover,  $Q_m, Q_{\bar{m}}$  are vectors on the  $\sigma$ -model of the vector-tensor scalars which can be thought as coming from the Kähler connection of special geometry after dualization.

As far as the field strengths appearing in the fermions transformation laws are concerned, we find the following relations:<sup>12</sup>

$$F_{ab}^\Lambda = 2 \left( f_i^\Lambda G_{ab}^{i-} + \bar{f}_{\bar{i}}^\Lambda G_{ab}^{\bar{i}+} \right) + i \left( L^\Lambda T_{ab}^+ + \bar{L}^\Lambda T_{ab}^- \right) \quad (4.17)$$

$$F_{M|ab} = 2 \left( g_{Mi} G_{ab}^{i-} + \bar{g}_{M\bar{i}} G_{ab}^{\bar{i}+} \right) + i \left( M_M T_{ab}^+ + \bar{M}_M T_{ab}^- \right) \quad (4.18)$$

$$h_{Mi} G_{ab}^{i-} = 0 \quad (4.19)$$

$$H_{Mabc} = -\frac{i}{3} \epsilon_{abcd} \left[ (h_{Mi} D^d z^i - h_{M\bar{i}} D^d \bar{z}^{\bar{i}}) \right]. \quad (4.20)$$

Finally, on the gauge sector we obtain a set of relations involving the Killing vectors and the fermionic shifts. They can be split into an  $SU(2)$ -singlet sector and an  $SU(2)$ -adjoint sector, corresponding to the  $U(1)$  and  $SU(2)$  parts of the R-symmetry. Indeed, as we will see in the discussion of the scalar potential, the  $U(1)$  part is related to the vector and vector-tensor couplings, while the  $SU(2)$ -part pertains to the hypermultiplets and scalar-tensor multiplets.

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<sup>12</sup>We recall that in the superspace rheonomic approach the components of the curvatures along the bosonic vielbein  $V^a$  do not coincide with their space-time components along the differentials  $dx^\mu$ . Actually, they differ from the space-time components by fermion bilinears and they coincide, in the component approach, with the supercovariant field strengths. The fermion bilinears are immediately retrieved from the superspace parametrizations given in Appendix B.1 by projecting the supercurvatures  $H_{Mabc}$ ,  $F_{ab}^\Lambda$ ,  $F_{Mab}$ ,  $Z_a^i$ , on the space-time differentials (see, for example, Appendix A of ref. [28].).

**$U(1)$ -sector relations:**

$$k_{\Lambda}^i L^{\Lambda} - k^{iN} M_N = 0; \quad (4.21)$$

$$W^{i[AB]} = \epsilon^{AB} \left( k_{\Lambda}^i \bar{L}^{\Lambda} - k^{iM} \bar{M}_M \right); \quad (4.22)$$

$$Q_m W^{m[AB]} = Q_{\bar{m}} \bar{W}_{[AB]}^{\bar{m}} = 0; \quad (4.23)$$

$$g_{Mi} \epsilon_{AB} W^{iAB} + \bar{g}_{M\bar{i}} \epsilon^{AB} \bar{W}_{AB}^{\bar{i}} = 0; \quad (4.24)$$

$$P_M m^{\Lambda M} = -\frac{1}{2} f_i^{\Lambda} \epsilon_{AB} W^{iAB} - \frac{1}{2} \bar{f}_{\bar{i}}^{\Lambda} \epsilon^{AB} \bar{W}_{AB}^{\bar{i}}. \quad (4.25)$$

We note that eq. (4.21) differs from the analogous one for special geometry by the presence on the right-hand side of the magnetic Killing vector  $k^{Mn}$ . Furthermore, as eq. (4.22) shows, the singlet shift of the spinors in the vector-tensor multiplets is symplectic invariant, once the gauging of the vector multiplets and of the graviphoton is turned off.

**$SU(2)$ -sector relations:**

$$P_M S^{AB} = h_{Mi} W^{i(AB)}; \quad (4.26)$$

$$\epsilon_{AC} \left( f_i^{\Lambda} W^{i(CB)} - 2L^{\Lambda} \bar{S}^{CB} \right) + \epsilon^{BC} \left( \bar{f}_{\bar{i}}^{\Lambda} \bar{W}_{(AC)}^{\bar{i}} - 2\bar{L}^{\Lambda} S_{AC} \right) = 0; \quad (4.27)$$

$$\epsilon_{AC} \left( g_{Mi} W^{i(CB)} - 2M_M \bar{S}^{CB} \right) + \epsilon^{BC} \left( \bar{g}_{M\bar{i}} \bar{W}_{(AC)}^{\bar{i}} - 2\bar{M}_M S_{AC} \right) = 0. \quad (4.28)$$

We observe that these relations are simple extensions of those obtained for standard  $D = 4$ ,  $N = 2$  supergravity.

From the physical point of view the main interest is of course in the supersymmetry transformation laws, which are an immediate consequence of the solution for the curvatures given in Appendix B.1. Up to 3-fermions<sup>13</sup> they read:

$$\delta V_{\mu}^a = -i \bar{\psi}_{A\mu} \gamma^a \epsilon^A - i \bar{\psi}_{\mu}^A \gamma^a \epsilon_A \quad (4.29)$$

$$\begin{aligned} \delta B_{M\mu\nu} = & h_{Mi} \bar{\epsilon}_A \gamma_{\mu\nu} \lambda^{iA} + \bar{h}_{M\bar{i}} \bar{\epsilon}^A \gamma_{\mu\nu} \lambda_{\bar{A}}^{\bar{i}} - i P_M \left( \bar{\epsilon}_A \gamma_{[\mu} \psi_{\nu]}^A - \bar{\psi}_{A[\mu} \gamma_{\nu]} \epsilon^A \right) + \\ & + 2 \left( d_{\Lambda\Sigma M} A_{[\mu}^{\Sigma} + \hat{T}_{\Lambda M}^N A_{N[\mu} \right) \left( L^{\Lambda} \bar{\epsilon}^A \psi_{\nu]}^B \epsilon_{AB} + \bar{L}^{\Lambda} \bar{\epsilon}_A \psi_{B\nu]} \epsilon^{AB} \right) \end{aligned} \quad (4.30)$$

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<sup>13</sup>In eqs. (4.35) and (4.36) we have kept the 3-fermions terms of type  $\epsilon\psi\chi$  since they are essential in the analysis of the gauge fermion shifts.

$$\begin{aligned}\delta A_\mu^\Lambda &= 2L^\Lambda \bar{\psi}_\mu^A \epsilon^B \epsilon_{AB} + 2\bar{L}^\Lambda \bar{\psi}_{A\mu} \epsilon_B \epsilon^{AB} + i f_i^\Lambda \bar{\lambda}^{iA} \gamma_\mu \epsilon^B \epsilon_{AB} + \\ &\quad + i \bar{f}_i^\Lambda \bar{\lambda}^{i\bar{A}} \gamma_\mu \epsilon_B \epsilon^{AB}\end{aligned}\tag{4.31}$$

$$\begin{aligned}\delta A_{M\mu} &= 2M_M \bar{\psi}_\mu^A \epsilon^B \epsilon_{AB} + 2\bar{M}_M \bar{\psi}_{A\mu} \epsilon_B \epsilon^{AB} + i g_{Mi} \bar{\lambda}^{iA} \gamma_\mu \epsilon^B \epsilon_{AB} + \\ &\quad + i \bar{g}_{M\bar{i}} \bar{\lambda}^{i\bar{A}} \gamma_\mu \epsilon_B \epsilon^{AB}\end{aligned}\tag{4.32}$$

$$\delta z^i = \bar{\epsilon}_A \lambda^{iA}\tag{4.33}$$

$$\delta \bar{z}^{\bar{i}} = \bar{\epsilon}^A \lambda_{\bar{A}}^{\bar{i}}\tag{4.34}$$

$$\begin{aligned}\delta \psi_{A\mu} &= D_\mu \epsilon_A + \epsilon_{AB} T_{\mu\nu}^- \gamma^\nu \epsilon^B + h_\mu \epsilon_A + i S_{AB} \gamma_\mu \epsilon^B + \\ &\quad + \frac{i}{2} \epsilon_A \left( Q_m \bar{\psi}_{B\mu} \lambda^{mB} + Q_{\bar{m}} \bar{\psi}_\mu^B \lambda_{\bar{B}}^{\bar{m}} \right) - \frac{i}{2} \psi_{A|\mu} \left( Q_m \bar{\epsilon}_B \lambda^{mB} + Q_{\bar{m}} \bar{\epsilon}^B \lambda_{\bar{B}}^{\bar{m}} \right)\end{aligned}\tag{4.35}$$

$$\begin{aligned}\delta \psi_\mu^A &= D_\mu \epsilon^A + \epsilon^{AB} T_{\mu\nu}^+ \gamma^\nu \epsilon_B - h_\mu \epsilon^A + i \bar{S}^{AB} \gamma_\mu \epsilon_B + \\ &\quad - \frac{i}{2} \epsilon_A \left( Q_m \bar{\psi}_{B\mu} \lambda^{mB} + Q_{\bar{m}} \bar{\psi}_\mu^B \lambda_{\bar{B}}^{\bar{m}} \right) + \frac{i}{2} \psi_{A|\mu} \left( Q_m \bar{\epsilon}_B \lambda^{mB} + Q_{\bar{m}} \bar{\epsilon}^B \lambda_{\bar{B}}^{\bar{m}} \right)\end{aligned}\tag{4.36}$$

$$\delta \lambda^{iA} = i D_\mu z^i \gamma^\mu \epsilon^A + G_{\mu\nu}^{i-} \gamma^{\mu\nu} \epsilon^{AB} \epsilon_B + W^{iAB} \epsilon_B\tag{4.37}$$

$$\delta \lambda_{\bar{A}}^{\bar{i}} = i D_\mu \bar{z}^{\bar{i}} \gamma^\mu \epsilon_A + G_{\mu\nu}^{\bar{i}+} \gamma^{\mu\nu} \epsilon_{AB} \epsilon^B + \bar{W}_{AB}^{\bar{i}} \epsilon^B\tag{4.38}$$

## 4.1 The scalar potential

To retrieve the equations of motion and in particular the scalar potential from the solution of Bianchi identities would be a straightforward but very cumbersome computation. They can be derived in a much easier way from the Lagrangian, which is presently under construction. We recall that in our complex formalism the dualization equations relating the antisymmetric tensors to the imaginary part of the scalars in the vector-tensor sector are a consequence of the Bianchi identities, and therefore they are valid only on-shell. Hence, prior to the explicit solution of the dualization equations, the scalar potential has formally exactly the same structure as in standard  $N = 2$  theory, and it can be computed from the fermionic shifts appearing in the supersymmetry transformation laws of the fermions (4.35) - (4.38), namely:

$$V \delta_A^C = -12 \bar{S}^{BC} S_{AB} + g_{i\bar{j}} W^{iBC} \bar{W}_{BA}^{\bar{j}}.\tag{4.39}$$

The fermion shifts can be read from the set of constraints on the gauge sector (4.21) - (4.28). In the absence of hypermultiplets (and leaving aside

possible Fayet–Iliopoulos terms) they are:

$$W^{i[AB]} = \epsilon^{AB} \left( k_{\Lambda}^i \bar{L}^{\Lambda} - k^{iM} \bar{M}_M \right) \quad (4.40)$$

$$S_{AB} = 0; \quad W^{i(AB)} = 0. \quad (4.41)$$

The above equations show that the only contributions to the scalar potential coming from the vector-tensor sector are singlets of  $SU(2)$  that belong to the  $U(1)$  part of the R-symmetry. Note that the contribution from the vector-tensor sector in  $W^{i[AB]}$ , in the absence of gauged isometries in the  $G_0$  directions (that is if  $k_{\Lambda}^m = \delta_{\Lambda}^M k_M^m$ ), is symplectic invariant (recalling that  $k^{iM} = \delta_m^i k^{mM}$ ).

In the standard  $N = 2$  supergravity, the  $SU(2)$  R-symmetry contribution to the scalar potential comes from the hypermultiplet sector, where the  $SU(2)$  is explicitly realized being part of the holonomy group of the quaternionic manifold. The  $SU(2)$  R-symmetry remains manifest when one considers the  $N = 2$ ,  $D = 4$  supergravity coupled to scalar-tensor multiplets [4, 5]<sup>14</sup>. The relevant point is that the contribution of that sector to the scalar potential is symplectic invariant.

We thus arrive at the important conclusion that  $D = 4$ ,  $N = 2$  supergravity coupled to scalar-tensor and vector-tensor multiplets only, has a symplectic invariant scalar potential<sup>15</sup>.

We notice however that the study of the minima of the scalar potential requires an explicit form of the geometry of the  $\sigma$ -model. This form has been computed explicitly in Appendix C by explicit dualization of the Kähler  $\sigma$ -model of a subsector of special geometry. Such dualization, however, is a consequence of Bianchi identities and therefore it is valid only on-shell. This seems to be a peculiarity of the formulation of  $N = 2$  supergravity coupled to vector-tensor multiplets. In absence of a factorization of the two  $\sigma$ -models of vector and vector-tensor multiplets, the explicit dualization gives an on-shell geometry whose metric is not hermitian, even in the subsector pertaining to undualized vector multiplets, as it is apparent from Appendix C.

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<sup>14</sup>This is obvious, since the scalar-tensor multiplet can be thought as resulting from dualization of (a subset of) the scalars in the hypermultiplet sector.

<sup>15</sup>Assuming that there are no gauged isometries in the graviphoton direction.

## 5 Some comments on $D = 5$ , $N = 2$ supergravity

The rich gauge structure underlying the bosonic sector of all FDA's can be in particular applied also to the  $N = 2$ ,  $D = 5$  case. When this theory is constructed in the framework of FDA's with the rheonomic approach, the final outcome is completely equivalent to the existing formulation in the literature [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. However, there are some features of the theory which are best appreciated in the light of the FDA approach, on which we would like to comment. Such features concern in particular the Higgs mechanism, since in our approach the 2-forms we start from are massless from the beginning. Let us then discuss how the results in the existing literature may be retrieved by starting with massless tensors<sup>16</sup>.

Actually, in our approach the mechanism for which the 2-forms become massive is left to the dynamics of the Lagrangian (or alternatively, in the supersymmetric case, also of the supersymmetric Bianchi identities). This is implemented via the Higgs mechanism. Even if this procedure is very well understood at the bosonic level, to implement it within a supersymmetric theory in  $D = 5$  is a non trivial task. This is due to the fact that the supersymmetry constraints require the vectors  $A^M$  giving mass to the tensors (in the notations of section 2) to be related to the tensors themselves in a non local way, involving Hodge-duality. This relation is codified in the so-called “*self-duality-in-odd-dimensions*” condition to which all the tensor fields in odd-dimensional supergravity theories have to comply [30]:

$$m^{MN} H_{N|abc} \propto \epsilon_{abcde} F^{M|de}. \quad (5.1)$$

In particular, for the five dimensional case the tensors are further required to be in even number.

Actually, in the approach currently adopted in the literature [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], the tensors  $B_M$  in the tensor multiplets are taken to be massive (and constrained to satisfy (5.1)) from the very beginning, without any tensor-gauge freedom.

To implement the Higgs mechanism on 2-forms at the supersymmetric level one could think of directly supersymmetrizing the FDA (2.19), and try

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<sup>16</sup>The explicit construction of the theory within the present approach has been in fact given in [29] (unpublished).

to give mass to the whole tensor multiplets by coupling them to  $n_T$  extra abelian vector multiplets added to the theory:

$$(A_\mu^M, \chi^{MA}, \phi^M), \quad (5.2)$$

where the vectors  $A^M$  and the tensors  $B_M$  admit the couplings and gauge invariance as in (2.19) and (2.22). If this would be the case, in the interacting theory the fields in the extra vector multiplets would couple to the tensor multiplets and one would end up with  $n_T$  *long* massive multiplets. We found, however, from explicit calculation that this is not the case, since supersymmetry transformations never relate the tensors  $B_M$  to the spinors  $\chi^{MA}$  nor to the scalars  $\phi^M$  in (5.2). Then the only way compatible with supersymmetry to couple  $N = 2$  supergravity with  $n_T$  massive tensors involves *short BPS* tensor multiplets

$$(B_{M|\mu\nu}, \lambda^{MA}, \varphi^M)$$

where the massive tensors  $B_M$  (that are complex, and hence in even number, because of CPT invariance of the BPS multiplet) have to satisfy (5.1). This is evident for the models having a six dimensional uplift, where the mass of the tensors is the BPS central charge gauged by the graviphoton  $g_{\mu 5}$  [31]. To show this, let us look at the subclass of models obtained by Scherk–Schwarz dimensional reduction from six dimensions. Indeed, the six-dimensional Lorentz algebra admits as irreducible representations self-dual tensors, satisfying

$$\partial_{[\hat{\mu}} B_{\hat{\nu}\hat{\rho}]M} = \frac{1}{6} \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}\hat{\tau}} \partial^{\hat{\sigma}} B_M^{\hat{\lambda}\hat{\tau}}, \quad \hat{\mu}, \hat{\nu}, \dots = 0, 1, \dots, 5. \quad (5.3)$$

Since  $N = 2$  matter-coupled supergravity in six dimensions contains one antiself-dual and  $n_T$  self-dual tensors in the vector representation of  $SO(1, n_T)$ , one can use the  $SO(n_T) \subset SO(1, n_T)$  global symmetry of the model to dimensionally reduce the theory on a circle down to five dimensions à la Scherk–Schwarz [31], with S-S phase  $m^{MN} = -m^{NM} \in SO(n_T)$ :

$$B_{\hat{\mu}\hat{\nu}M}(x, y_5) = \left( \exp[my_5] \right)_M^N \sum_n B_{\hat{\mu}\hat{\nu}N}^{(n)}(x) \exp \left[ \frac{in}{2\pi R} y_5 \right]. \quad (5.4)$$

Applying (5.4) to the self-duality relation (5.3) for the zero-mode, we find

$$\partial_{[\mu} B_{\nu\rho]M} = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma\lambda 5} (m_M^N B_N^{\sigma\lambda} + 2F_M^{\sigma\lambda}), \quad \mu = 0, 1, \dots, 4 \quad (5.5)$$

where  $F_{\sigma\lambda N} \equiv \partial_{[\sigma} B_{\lambda]5N}$ . Eq. (5.5) expresses the self-duality obeyed by the tensors in five dimensional supergravity. However, it also shows that the field-strengths of the vectors  $B_{\mu 5N}$ , that give mass to the tensors  $B_{\mu\nu M}$  via the anti-Higgs mechanism, are in fact the Hodge-dual of the tensors  $B_{\mu\nu M}$  themselves. From our analysis applied to  $N = 2$  supergravity in five dimensions, we find this to be a general fact, not necessarily related to theories admitting a six dimensional uplift: in each case, the massive tensor fields belong to short representations of supersymmetry, and the dynamical interpretation of the mechanism giving mass to the tensors requires the coupling of the massless tensors to gauge vectors which are the Hodge-dual of the tensors themselves.

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## A Constraints on 3-form Bianchi identities

$$m^{\Lambda M} k_M^\alpha = 0 \tag{A.1}$$

$$f_{[\Lambda\Sigma}^\Delta f_{\Gamma\Delta]}^\Pi = 2m^{\Pi M} L_{\Lambda\Sigma\Gamma M} \tag{A.2}$$

$$f_{\Sigma\Gamma}^\Lambda m^{\Gamma M} = m^{\Lambda N} T_{\Sigma N}^M \tag{A.3}$$

$$T_{\Lambda M}^N = d_{\Lambda\Sigma M} m^{\Sigma N} + T_{\Lambda\alpha}^N k_M^\alpha \tag{A.4}$$

$$T_{[\Lambda|M}^P T_{\Sigma]P}^N = \frac{1}{2} f_{\Lambda\Sigma}^\Gamma T_{\Gamma M}^N + k_M^\alpha N_{\alpha\Lambda\Sigma}^N + 3L_{\Lambda\Sigma\Gamma M} m^{\Gamma N} \tag{A.5}$$

$$T_{\Lambda M}^{(N} m^{\Lambda|P)} = k_M^\alpha N_\alpha^{NP} \tag{A.6}$$

$$T_{\Lambda M}{}^N k_N^\alpha = k_M^\beta T_{\Lambda\beta}{}^\alpha \quad (\text{A.7})$$

$$T_{[\Lambda|M}{}^N L_{\Sigma\Gamma\Delta]N} = k_M^\alpha M_{\Lambda\Sigma\Gamma\Delta\alpha} + \frac{3}{2} f_{[\Lambda\Sigma}{}^\Omega L_{\Gamma\Delta]\Omega M} \quad (\text{A.8})$$

$$T_{[\Lambda|M}{}^N d_{\Gamma|\Sigma]N} = \hat{f}_{\Lambda\Gamma}{}^\Pi d_{\Pi|\Sigma]M} + \frac{1}{2} f_{\Lambda\Sigma}{}^\Pi d_{\Gamma\Pi M} + k_M^\alpha L_{\Lambda\Sigma\Gamma\alpha} - 3L_{\Lambda\Sigma\Gamma M} \quad (\text{A.9})$$

$$T_{\Lambda\alpha}{}^\beta = T_{\Lambda\alpha}{}^M k_M^\beta \quad (\text{A.10})$$

$$T_{[\Lambda|\alpha}{}^\gamma T_{\Sigma]\gamma}{}^\beta = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma T_{\Gamma\alpha}{}^\beta + k_M^\beta N_{\alpha\Lambda\Sigma}{}^M \quad (\text{A.11})$$

$$T_{\Lambda\alpha}{}^\beta m^{\Lambda M} = -2N_\alpha{}^{MN} k_N^\beta \quad (\text{A.12})$$

$$T_{\Lambda\alpha}{}^\beta T_{\Sigma\beta}{}^M = \hat{f}_{\Lambda\Sigma}{}^\Pi T_{\Pi\alpha}{}^M + T_{\Lambda\alpha}{}^N T_{\Sigma N}{}^M - 2L_{\Lambda\Pi\Gamma\alpha} m^{\Pi M} + \\ + 2N_{\alpha\Lambda\Sigma}{}^M + 2N_\alpha{}^{MN} d_{\Sigma\Lambda N} \quad (\text{A.13})$$

$$T_{[\Lambda|\alpha}{}^\beta T'_{\Sigma]\beta}{}^M = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma T'_{\Gamma\alpha}{}^M - T'_{[\Lambda|\alpha}{}^N T_{\Sigma]N}{}^{(H)M} - L_{\Lambda\Sigma\Gamma\alpha} m^{\Gamma M} - N_{\alpha\Lambda\Sigma}{}^M \quad (\text{A.14})$$

$$T_{[\Lambda|\alpha}{}^\beta L_{\Sigma\Gamma]\Delta\beta} = T_{\Delta\alpha}{}^M L_{\Lambda\Sigma\Gamma M} - f_{[\Lambda\Sigma}{}^\Pi L_{\Gamma]\Pi\Delta\alpha} + L_{[\Lambda\Sigma|\Pi\alpha} \hat{f}_{\Gamma]\Delta}{}^\Pi + \\ + 4M_{\Lambda\Sigma\Gamma\Delta\alpha} + N_{\alpha[\Lambda\Sigma}{}^M d_{\Delta]\Gamma]M} \quad (\text{A.15})$$

$$T_{[\Lambda|\alpha}{}^\beta M_{\Sigma\Gamma\Delta\Pi]\beta} = 2M_{\Omega[\Lambda\Sigma\Gamma\alpha} f_{\Delta\Pi]}{}^\Omega + N_{\alpha[\Lambda\Sigma}{}^M L_{\Gamma\Delta\Pi]M} \quad (\text{A.16})$$

$$T_{[\Lambda|\alpha}{}^\beta N_{\beta|\Sigma\Gamma]}{}^M = -4M_{\Lambda\Sigma\Gamma\Delta\alpha} m^{\Delta M} - N_{\alpha[\Lambda|\Pi}{}^M f_{\Sigma\Gamma]}{}^\Pi + N_{\alpha[\Lambda\Sigma}{}^N T_{\Gamma]N}{}^M + \\ + 2N_\alpha{}^{MN} L_{\Lambda\Sigma\Gamma N} \quad (\text{A.17})$$

$$T_{\Lambda\alpha}{}^\beta N_\beta{}^{MN} = -2N_{\alpha\Lambda\Sigma}{}^{(M} m^{\Sigma|N)} + 2N_\alpha{}^{P(M} T_{\Lambda P}{}^{N)} \quad (\text{A.18})$$

$$T_{\Lambda\alpha}{}^M m^{\Lambda N} = T'_{\Lambda\alpha}{}^N m^{\Lambda M} - 2N_\alpha{}^{MN} \quad (\text{A.19})$$

$$T_{(\Lambda|\alpha}{}^M d_{\Sigma)\Gamma M} = -T'_{\Gamma\alpha}{}^M d_{(\Lambda\Sigma)M} - 2L_{\Gamma(\Lambda\Sigma)\alpha} + R_{\beta\Lambda\Sigma}(T_{\Gamma\alpha}{}^\beta + T'_{\alpha\Gamma}{}^M k_M^\beta) + \\ + 2R_{\alpha\Pi(\Lambda} \hat{f}_{\Gamma|\Sigma]}{}^\Pi \quad (\text{A.20})$$

## B Solution of superspace Bianchi identities for $D = 4$ , $N = 2$ vector-tensor supergravity

Differentiating eqs. (4.1) - (4.9) we get the supersymmetric BI's. Using the definitions:

$$DF^\Lambda = dF^\Lambda + \hat{f}_{\Sigma\Gamma}{}^\Lambda A^\Sigma F^\Gamma + m^{\Lambda N} \hat{T}_{\Sigma N}{}^M A_M F^\Sigma \quad (\text{B.1})$$

$$DF_M = dF_M + \hat{T}_{\Lambda M}{}^N (A^\Lambda F_N - A_N F^\Lambda) \quad (\text{B.2})$$



$$DH_M = dH_M + \hat{T}_{\Lambda M}^N A^\Lambda H_N \quad (B.3)$$

$$DL^\Lambda = dL^\Lambda + \hat{f}_{\Sigma\Gamma}^\Lambda A^\Sigma L^\Gamma + m^{\Lambda N} \hat{T}_{\Sigma N}^M A_M L^\Sigma + i\mathcal{Q}L^\Lambda \quad (B.4)$$

$$DM_M = dM_M + \hat{T}_{\Lambda M}^N (A^\Lambda M_N - A_N L^\Lambda) + i\mathcal{Q}M_M \quad (B.5)$$

$$DP_M = dP_M + \hat{T}_{\Lambda M}^N A^\Lambda P_N \quad (B.6)$$

for the covariant derivatives of gauge-covariant quantities, where the gauged U(1) connection  $\mathcal{Q}$  has non vanishing components only along the  $dz^x$  and  $d\bar{z}^{\bar{x}}$  differentials, the Bianchi identities read:

$$D\mathcal{R}^a_b = 0 \quad (B.7)$$

$$DT^a + \mathcal{R}^a_b V^b - i\bar{\psi}^A \gamma^a \rho_A + i\bar{\psi}_A \gamma^a \rho^A = 0 \quad (B.8)$$

$$\nabla \rho_A + \frac{1}{4} \gamma_{ab} \mathcal{R}^{ab} \psi_A - \frac{i}{2} K \psi_A = 0 \quad (B.9)$$

$$\nabla \rho^A + \frac{1}{4} \gamma_{ab} \mathcal{R}^{ab} \psi^A + \frac{i}{2} K \psi^A = 0 \quad (B.10)$$

$$DF^\Lambda = DL^\Lambda \bar{\psi}^A \psi^B \epsilon_{AB} + D\bar{L}^\Lambda \bar{\psi}_A \psi_B \epsilon^{AB} - 2L^\Lambda \bar{\psi}^A \rho^B \epsilon_{AB} + \\ - 2\bar{L}^\Lambda \bar{\psi}_A \rho_B \epsilon^{AB} + m^{\Lambda M} (H_M - iP_M \bar{\psi}_A \gamma_a \psi^A V^a) \quad (B.11)$$

$$DF_M = DM_M \bar{\psi}^A \psi^B \epsilon_{AB} + D\bar{M}_M \bar{\psi}_A \psi_B \epsilon^{AB} - 2M_M \bar{\psi}^A \rho^B \epsilon_{AB} + \\ - 2\bar{M}_M \bar{\psi}_A \rho_B \epsilon^{AB} \quad (B.12)$$

$$DH_M = iDP_M \bar{\psi}_A \gamma_a \psi^A V^a - iP_M \left( \bar{\psi}_A \gamma_a \rho^A + \bar{\psi}^A \gamma_a \rho_A \right) V^a + \\ + P_M \bar{\psi}_A \gamma_a \psi^A \bar{\psi}_B \gamma^a \psi^B + \\ + \left[ d_{\Lambda\Sigma M} \left( F^\Sigma - L^\Sigma \bar{\psi}^A \psi^B \epsilon_{AB} - \bar{L}^\Sigma \bar{\psi}_A \psi_B \epsilon^{AB} \right) + \right. \\ \left. + \hat{T}_{\Lambda M}^N \left( F_N - M_N \bar{\psi}^A \psi^B \epsilon_{AB} - \bar{M}_N \bar{\psi}_A \psi_B \epsilon^{AB} \right) \right] \cdot \\ \cdot \left( F^\Lambda - L^\Lambda \bar{\psi}^C \psi^D \epsilon_{CD} - \bar{L}^\Lambda \bar{\psi}_C \psi_D \epsilon^{CD} \right) \quad (B.13)$$

$$D^2 z^i = k_\Lambda^i \left( F^\Lambda - L^\Lambda \bar{\psi}^A \psi^B \epsilon_{AB} - \bar{L}^\Lambda \bar{\psi}_A \psi_B \epsilon^{AB} \right) + \\ - k^{mN} \left( F_N - M_N \bar{\psi}^A \psi^B \epsilon_{AB} - \bar{M}_N \bar{\psi}_A \psi_B \epsilon^{AB} \right) \quad (B.14)$$

$$\nabla^2 \lambda^{iA} = \frac{1}{4} \gamma_{ab} \mathcal{R}^{ab} \lambda^{iA} + \frac{i}{2} K \lambda^{iA} + R^i_j \lambda^{jA} \quad (B.15)$$

$$\nabla^2 \bar{\lambda}^{\bar{i}}_A = \frac{1}{4} \gamma_{ab} \mathcal{R}^{ab} \bar{\lambda}^{\bar{i}}_A - \frac{i}{2} K \bar{\lambda}^{\bar{i}}_A + R^{\bar{i}}_{\bar{j}} \bar{\lambda}^{\bar{j}}_A \quad (B.16)$$

where:

$$K = d\mathcal{Q} \quad (\text{B.17})$$

is the gauged Kähler 2-form.

## B.1 Parametrization of the curvatures in $D = 4$ , $N = 2$ superspace

$$\mathcal{T}^a = 0 \quad (\text{B.18})$$

$$\begin{aligned} \rho_A = & \rho_{Aab} V^a V^b + \epsilon_{AB} T_{ab}^- \gamma^b \psi^B V^a + h_a \psi_A V^a + \\ & + \frac{i}{2} \psi_A \left( Q_m \bar{\psi}_B \lambda^{mB} + Q_{\bar{m}} \bar{\psi}^B \lambda_{\bar{m}}^{\bar{B}} \right) + i S_{AB} \gamma_a \psi^B V^a \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} \rho^A = & \rho_{ab}^A V^a V^b + \epsilon^{AB} T_{ab}^+ \gamma^b \psi_B V^a - h_a \psi^A V^a + \\ & - \frac{i}{2} \psi_A \left( Q_m \bar{\psi}_B \lambda^{mB} + Q_{\bar{m}} \bar{\psi}^B \lambda_{\bar{m}}^{\bar{B}} \right) + i \bar{S}^{AB} \gamma_a \psi_B V^a \end{aligned} \quad (\text{B.20})$$

$$H_M = \hat{H}_{M|abc} V^a V^b V^c + h_{Mi} \bar{\psi}_A \gamma_{ab} \lambda^{iA} V^a V^b + h_{M\bar{i}} \bar{\psi}^A \gamma_{ab} \lambda^{\bar{i}}_A V^a V^b \quad (\text{B.21})$$

$$F^\Lambda = \hat{F}_{ab}^\Lambda V^a V^b + i f_i^\Lambda \bar{\psi}^A \gamma_a \lambda^{iB} \epsilon_{AB} V^a + i \bar{f}_{\bar{i}}^\Lambda \bar{\psi}_A \gamma_a \lambda^{\bar{i}}_B \epsilon^{AB} V^a \quad (\text{B.22})$$

$$F_M = \hat{F}_{Mab} V^a V^b + i g_{Mi} \bar{\psi}^A \gamma_a \lambda^{iB} \epsilon_{AB} V^a + i \bar{g}_{M\bar{i}} \bar{\psi}_A \gamma_a \lambda^{\bar{i}}_B \epsilon^{AB} V^a \quad (\text{B.23})$$

$$Dz^i = D_a z^i V^a + \bar{\psi}_A \lambda^{iA} \quad (\text{B.24})$$

$$D\bar{z}^{\bar{i}} = D_a \bar{z}^{\bar{i}} V^a + \bar{\psi}^A \lambda^{\bar{i}}_A \quad (\text{B.25})$$

$$\nabla \lambda^{iA} = \hat{\nabla}_a \lambda^{iA} V^a + i D_a z^i \gamma^a \psi^A + G_{ab}^{i-} \gamma^{ab} \epsilon^{AB} \psi_B + W^{iAB} \psi_B \quad (\text{B.26})$$

$$\nabla \lambda^{\bar{i}}_A = \hat{\nabla}_a \lambda^{\bar{i}}_A V^a + i D_a \bar{z}^{\bar{i}} \gamma^a \psi_A + G_{ab}^{\bar{i}+} \gamma^{ab} \epsilon_{AB} \psi^B + \bar{W}^{\bar{i}}_{AB} \psi^B \quad (\text{B.27})$$

## C The vector-tensor $\sigma$ -model metric

Let us start from the (ungauged) kinetic term of special geometry:

$$\begin{aligned} \mathcal{L}_{kin} &= g_{i\bar{j}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} = \\ &= g_{r\bar{s}} \partial_\mu z^r \partial^\mu \bar{z}^{\bar{s}} + g_{m\bar{n}} \partial_\mu z^m \partial^\mu \bar{z}^{\bar{n}} + \\ &\quad + g_{r\bar{n}} \partial_\mu z^r \partial^\mu \bar{z}^{\bar{n}} + g_{m\bar{s}} \partial_\mu z^m \partial^\mu \bar{z}^{\bar{s}}. \end{aligned} \quad (\text{C.1})$$

where we have denoted by  $r, s$  the indices of the scalar fields of the vector multiplets which will not undergo the dualization. Decomposing the differentials into real and imaginary parts  $dz^i = dx^i + idy^i$ , we easily get:

$$\mathcal{L}_{kin} = D_{ij} (\partial_\mu x^i \partial^\mu x^j + \partial_\mu y^i \partial^\mu y^j) + 2\Gamma_{ij} \partial_\mu x^i \partial^\mu y^j, \quad (\text{C.2})$$

where

$$D_{ij} = \frac{1}{2}(g_{i\bar{j}} + g_{\bar{i}j}) \quad ; \quad \Gamma_{ij} = -\frac{i}{2}(g_{i\bar{j}} - g_{\bar{i}j}). \quad (\text{C.3})$$

To perform the dualization on the vector-tensor multiplet sector, we introduce the Lagrange multiplier  $Y_\mu^M$  by the substitution  $\partial_\mu y^m \Rightarrow \partial_\mu y^M \equiv Y_\mu^M$  and add to the Lagrangian (C.2) the term  $\frac{1}{3!} Y_\mu^M H_{M\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$ . Varying then the new Lagrangian with respect to  $Y_\mu^M$  one obtains

$$Y_\mu^M = -\frac{1}{2} D^{MN} \left( 2\Gamma_{iN} \partial_\mu x^i + 2D_{rN} \partial_\mu y^r + \frac{1}{3!} H_N^{\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \right), \quad (\text{C.4})$$

where  $D^{MN}$  is the inverse matrix of  $D_{MN}$ . Substituting (C.4) in (C.2) one obtains the dual Lagrangian, namely:

$$\begin{aligned} \mathcal{L}_{dual} = & \Delta_{ij} \partial_\mu x^i \partial^\mu x^j + \tilde{\Delta}_{rs} \partial_\mu y^r \partial^\mu y^s + 2\tilde{\Gamma}_{ir} \partial_\mu x^i \partial^\mu y^r + \\ & -\frac{1}{3!} D^{MN} H_{M\nu\rho\sigma} \left[ \frac{1}{4} H_N^{\nu\rho\sigma} + (\Gamma_{iN} \partial_\mu x^i + D_{rN} \partial_\mu y^r) \epsilon^{\mu\nu\rho\sigma} \right] \end{aligned} \quad (\text{C.5})$$

where

$$\Delta_{ij} = D_{ij} + \Gamma_{iM} D^{MN} \Gamma_{Nj} \quad (\text{C.6})$$

$$\tilde{\Delta}_{rs} = D_{rs} - D_{rM} D^{MN} D_{Ns} \quad (\text{C.7})$$

$$\tilde{\Gamma}_{ir} = D_{ir} - \Gamma_{iM} D^{MN} D_{Nr}. \quad (\text{C.8})$$

give the  $\sigma$ -model metric after dualization.

## D Some notations for the four dimensional theories

We use throughout the paper a mostly minus space-time metric  $\eta_{ab}$ .

The  $\gamma_5$  matrix is defined as

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3. \quad (\text{D.1})$$

Given an (anti-)self-dual tensor  $F^\pm$ , the following relations are true:

$$*F^\pm = \mp iF^\pm \quad (\text{D.2})$$

Furthermore, we use the following Fierz identities among spinor 1-forms:

$$\psi_A \bar{\psi}_B = \frac{1}{2} \bar{\psi}_B \psi_A - \frac{1}{8} \gamma_{ab} \bar{\psi}_B \gamma^{ab} \psi_A \quad (\text{D.3})$$

$$\psi_A \bar{\psi}^B = \frac{1}{2} \gamma_a \bar{\psi}^B \gamma^a \psi_A. \quad (\text{D.4})$$

In terms of irreducible spinor representations, the following 3-gravitino relations hold:

$$\psi_A \bar{\psi}_B \psi_C = \frac{1}{4} \epsilon_{BC} (\Theta_A^+ - i \Theta_A^-) \quad (\text{D.5})$$

$$\psi_A \bar{\psi}^B \psi^C = \frac{1}{4} \epsilon^{BC} (\Theta_A^+ + i \Theta_A^-) \quad (\text{D.6})$$

$$\gamma_{ab} \psi_A \bar{\psi}_B \gamma^{ab} \psi_C = 2\epsilon_{A(B} (\Theta_{C)}^+ - i \Theta_{C)}^-) \quad (\text{D.7})$$

$$\gamma_{ab} \psi^A \bar{\psi}^B \gamma^{ab} \psi^C = 2\epsilon^{A(B} (\Theta^{+C)} - i \Theta^{-C}) \quad (\text{D.8})$$

$$\gamma_a \psi^A \bar{\psi}_B \gamma^a \psi^B = -\frac{1}{2} \epsilon^{AC} (\Theta_C^+ + i \Theta_C^-) \quad (\text{D.9})$$

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